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SMOOTH AND TIME-OPTIMAL TRAJECTORY PLANNING FOR INDUSTRIAL MANIPULATORS ALONG SPECIFIED PATHS

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ABSTRACT

This paper presents a method for determining smooth and time-optimal path-constrained trajectories for robotic manipulators. The desired smoothness of the trajectory is imposed through limits on the actuator jerks. The third derivative of the path parameter with respect to time, the pseudo-jerk, is the controlled input. The limits on the actuator torques translate into state-dependent limits on the pseudo-acceleration. The time-optimal control objective is cast as an optimization problem by using cubic splines to parameterize the state space trajectory. The optimization problem is solved using the flexible tolerance method.

INTRODUCTION

The need for increased productivity in path-following industrial robotic applications has been addressed in the literature by determining path-constrained time-optimal motions (PCTOM) while accounting for actuator torque limits; see (Bobrow, 1985)[1], (Pfeiffer, 1987)[9], (Shiller, 1992)[13]. In these formulations, the joint actuator torques are the controlled inputs and the open loop control schemes result in bang-bang or bang-singular-bang controls (Chen, 1989)[2].

PCTOM trajectories compute the maximum velocity achievable at the robot tip while still following the prescribed path. However, implementation of PCTOM in physical manipulators has drawbacks, such as joint oscillations due to finite joint stiffness and overshoot of the nominal torque limits due to unmodelled actuator dynamics. The resultant extra strain on the robot actuators could cause them to fail frequently (Li, 1997)[6], reducing the productivity of the entire workcell.

At the trajectory planning level, a number of different techniques have been devised to address the problem of discontinuous actuator torques. A modified cost function, such as time-joint torques (Pfeiffer, 1987)[9] or time-square of joint torques (Shiller, 1994)[11], smoothes the controls and helps to improve the tracking accuracy, at the expense of motion time.

Another way of smoothing the controls is to parameterize the path by using functions that are at least C^2 continuous. Cubic splines used for path parameterization with time as the cost function (Lin, 1983)[7] result in trajectories that have continuous joint accelerations. However, the limits on the joint variables are very conservative, since they need be constant over the entire space and, therefore, are chosen as the lowest maximum. Incorporating the actuator dynamics in this problem formulation (Tarkiainen, 1993)[14] transforms the actuator voltages into the limited controlled inputs. The optimal trajectory is bang-bang in the new controls and the actuator torques are no longer limited. Also, the case of singular controls is not considered since they can be avoided by an appropriate selection of the path (Shiller, 1992)[13] or by convexifying the set of admissible controls (Shiller, 1994)[10].

In this paper, a method is presented for determining time-optimal trajectories subject to actuator torque and jerk limits. The resulting trajectories will be called smooth path-constrained time-optimal motions (SPCTOM) to distinguish them from the path-constrained time-optimal motions (PCTOM), which do not consider jerk limits.

The actuator jerk limits are imposed in order to plan feasible optimal motions for industrial applications, which, more and more, use newer light direct drive manipulators. Moreover, in industrial applications, an accurate robot model is likely not readily available and the controller strategy cannot be modified by the user. In such cases, unlimited jerks can cause severe vibrations in the arm possibly leads to the failure of the actuators themselves. The SPCTOM trajectories filter the jerks from the trajectory at the planning level, thus leaving more authority to the tracking controller to compensate for disturbances. Potentially, they could also compensate for larger modelling errors.

Motion planning problems such as obstacle avoidance are beyond the scope of this paper, since the path is assumed to be preimposed. Path-constrained motions are specific to contour following applications. Typically, they are also used in point-to-point motion planning in cluttered environments (Shiller, 1989)[12], when the geometric constraints (obstacles, joint limits) are satisfied by the path planner and the dynamic constraints are left to the trajectory planner.

PROBLEM FORMULATION

The basic problem

The problem of smooth path-constrained time-optimal motion (SPCTOM) planning can be stated as follows:

$$\min_{\dot{\mathbf{T}} \in \mathbf{\Omega}} J = \int_0^{t_f} 1 dt, \tag{1}$$

subject to the manipulator dynamics:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathbf{T}}\mathbf{C}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{T}, \tag{2}$$

the boundary conditions:

$$\begin{aligned} \mathbf{q}(0) = \mathbf{q_0} \quad &; \quad \mathbf{q}(t_f) = \mathbf{q_f}\ ; \\ \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}(t_f) = 0 \quad &; \quad \ddot{\mathbf{q}}(0) = \ddot{\mathbf{q}}(t_f) = 0, \end{aligned} \tag{3}$$

the path constraints:

$$\mathbf{r} = \mathbf{r}(s), \tag{4}$$

the actuator torque limits:

$$\mathbf{T}_{min} \le \mathbf{T} \le \mathbf{T}_{max}, \tag{5}$$

and the actuator jerk limits:

$$\dot{\mathbf{T}}_{min} \le \dot{\mathbf{T}} \le \dot{\mathbf{T}}_{max}, \tag{6}$$

where n is the number of degrees of freedom of the manipulator. Furthermore, $\mathbf{q} \in \mathbf{R}^n$ is the vector of joint positions, $\mathbf{T} \in \mathbf{R}^n$ is the vector of actuator torques, $\dot{\mathbf{T}} \in \mathbf{R}^n$ is the vector of actuator jerks, $\mathbf{M}(\mathbf{q}) \in \mathbf{R}^{n\times n}$ is the inertia matrix of the manipulator, $\mathbf{C}(\mathbf{q}) \in \mathbf{R}^{n\times n\times n}$ is a third order tensor representing the coefficients of the centrifugal and Coriolis forces, $\mathbf{G}(\mathbf{q}) \in \mathbf{R}^n$ is the vector of gravity terms, and $\mathbf{r} \in \mathbf{R}^3$ is a C^1 continuous curve parametrized by s, which may be, for example, the arc length. To simplify the dynamics, viscous and static friction terms have been neglected.

In the above formulation, the actuator jerks represent the bounded controls. However, these controls would later have to be integrated in order to derive the actual system inputs in terms of desired actuator torques. Since the Lagrangian form of the robot dynamics incorporates only the actuator torques, the third order dynamics is required. Differentiation of (2) with respect to time results in:

$$\begin{aligned} \mathbf{M}(\mathbf{q})\dddot{\mathbf{q}} + \dot{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}} + \ddot{\mathbf{q}}^{\mathbf{T}}\mathbf{C}(\mathbf{q})\dot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathbf{T}}\dot{\mathbf{C}}(\mathbf{q})\dot{\mathbf{q}} + \\ \dot{\mathbf{q}}^{\mathbf{T}}\mathbf{C}(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{G}}(\mathbf{q}) = \dot{\mathbf{T}}. \end{aligned} \tag{7}$$

Equation (7) is taken as the dynamics of the system, with $\dot{\boldsymbol{\tau}}$ representing the n-dimensional bounded controls.

System dynamics for path-constrained motions

The dynamic system described by Equation (7) has $3n$ degrees of freedom. However, the path constraints (4) parameterize the tip position by a single parameter s and reduce the order of the system to 3. By expressing the joint positions, velocities, accelerations, and jerks as functions of the path parameter s, the actuator torque and jerk limits are transformed into limits on $\ddot{s}$, the pseudo-acceleration, and limits on the $\dddot{s}$, the pseudo-jerk, respectively:

$$\mathbf{T}_{min} \le \mathcal{A}(s)\ddot{s} + \mathcal{B}(s)\dot{s}^2 + \mathcal{C}(s) \le \mathbf{T}_{max} \tag{8}$$

$$\dot{\mathbf{T}}_{min} \le \mathbf{a}(s)\dddot{s} + \mathbf{b}(s)\dot{s}\ddot{s} + \mathbf{c}(s)\dot{s}^3 + \mathbf{d}(s)\dot{s} \le \dot{\mathbf{T}}_{max}, \tag{9}$$

where:

$$\mathcal{A}(s) = \mathbf{Mq}', \tag{10}$$
$$\mathcal{B}(s) = \mathbf{Mq}'' + \mathbf{q}^{'T}\mathbf{Cq}', \tag{11}$$
$$\mathcal{C}(s) = \mathbf{G}, \tag{12}$$
$$\mathbf{a}(s) = \mathbf{Mq}', \tag{13}$$
$$\mathbf{b}(s) = 3\mathbf{Mq}'' + \frac{d\mathbf{M}}{ds}\mathbf{q}' + 2\mathbf{q}^{'T}\mathbf{Cq}', \tag{14}$$
$$\mathbf{c}(s) = \mathbf{Mq}''' + \frac{d\mathbf{M}}{ds}\mathbf{q}'' + \mathbf{q}^{''T}\mathbf{Cq}' + \mathbf{q}^{'T}\frac{d\mathbf{C}}{ds}\mathbf{q}' + + \mathbf{q}^{'T}\mathbf{Cq}'', \tag{15}$$
$$\mathbf{d}(s) = \frac{d\mathbf{G}}{ds}\dot{s}, \tag{16}$$

with ($'$) denoting derivatives with respect to the path parameter.

The states of the reduced system are $\mathbf{x} = (s \quad \dot{s} \quad \ddot{s})^T$, while $\dddot{s}$ is the scalar control u. The SPCTOM planning problem is reformulated as:

$$\min_u J = \int_0^{t_f} 1dt, \tag{17}$$

subject to the system dynamics:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) = [x_2 \quad x_3 \quad u]^T, \tag{18}$$

the boundary conditions:

$$\mathbf{x}_0 = (s_0 \quad \dot{s}_0 \quad \ddot{s}_0)^T$$
$$\mathbf{x}_f = (s_f \quad \dot{s}_f \quad \ddot{s}_f)^T, \tag{19}$$

the state-dependent control constraints:

$$\mathbf{g}_1(\mathbf{x}, u) = \mathbf{a}\dddot{s} + \mathbf{b}\dot{s}\ddot{s} + \mathbf{c}\dot{s}^3 + \mathbf{d}\dot{s} - \dot{\mathbf{T}}_{max} \leq \mathbf{0} \tag{20}$$

$$\mathbf{g}_2(\mathbf{x}, u) = \dot{\mathbf{T}}_{min} - \mathbf{a}\dddot{s} - \mathbf{b}\dot{s}\ddot{s} - \mathbf{c}\dot{s}^3 - \mathbf{d}\dot{s} \leq \mathbf{0} \tag{21}$$

and the state inequality constraints:

$$\mathbf{g}_3(\mathbf{x}) = \mathcal{A}\ddot{s} + \mathcal{B}\dot{s}^2 + \mathcal{C} - \mathbf{T}_{max} \leq \mathbf{0} \tag{22}$$
$$\mathbf{g}_4(\mathbf{x}) = \mathbf{T}_{min} - \mathcal{A}\ddot{s} - \mathcal{B}\dot{s}^2 - \mathcal{C} \leq \mathbf{0} \tag{23}$$

SOLUTION OF THE SPCTOM

The above problem is a time-optimal control problem for a first order linear system with nonlinear state and control inequalities and preimposed initial and final states. Its solution is bang-bang or bang-singular-bang (Hocking, 1991)[5]. Similar two-point boundary value problems with bang-bang controls have been solved either by multiple shooting (Gourdeau, 1989)[3] or by a search for the switching points (Bobrow 1985)[1], (Pfeiffer, 1987)[9]. A multiple shooting method requires the number of switching points to be known *a priori*. However, this number is not known for the SPCTOM.

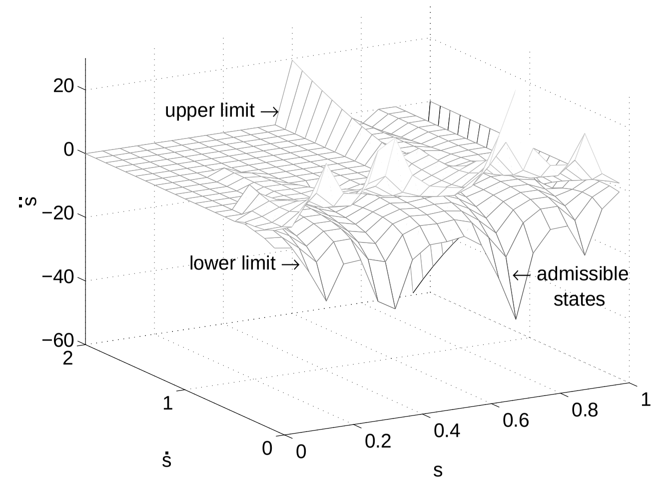


Figure 1: LIMITING SURFACES IN THE $s-\dot{s}-\ddot{s}$ PHASE SPACE.

On the other hand, a search for the switching points is impractical, since the state space is 3-dimensional. The search should be performed along the surfaces separating the admissible states from the inadmissible ones. An example of such surfaces is presented in Figure 1, where the top and bottom limiting surfaces are plotted and the feasible states lie in between. While potential switching points can be determined, they are not necessarily connected to each other. Also, for singular arcs, that lie entirely on one of the limiting surfaces, it is not clear in which direction the state should advance. This is opposed to the case of 2-dimensional state space, where one single possible direction exists, along the the limiting curve.

To resolve these difficulties, the SPCTOM is approximated herein in the s-$\dot{s}$ phase plane. The motivation is that in this plane both trajectory end-points are fixed, while in the time domain the final point is free. The motion time is computed as:

$$t(s) = \int_{s_0}^{s_f} \frac{ds}{\dot{s}}, \tag{24}$$

where s_0 and s_f are the initial and the final values of the path parameter, respectively. Therefore, the SPCTOM in the s-$\dot{s}$ phase plane is the smooth curve that minimizes $t(s)$

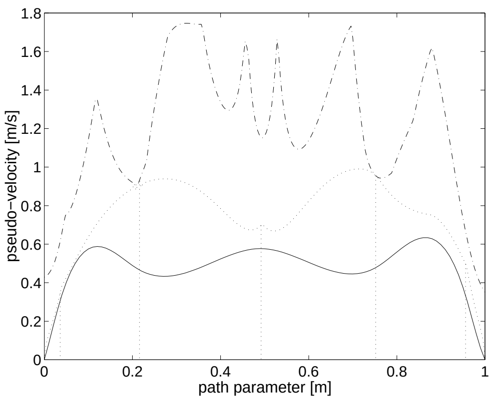


Figure 2: SWITCHING POINTS OF THE PCTOM (DOTTED LINE) AND A SAMPLE SPLINED TRAJECTORY (SOLID LINE). THE DASH-DOT LINE IS THE VELOCITY LIMIT CURVE WHEN TORQUE LIMITS ARE CONSIDERED.

over the curve while not violating actuator torque and/or actuator jerk limits.

In view of the above, the optimal motion is determined by an optimization of a base trajectory. A set of cubic splines with preselected knot-point locations are chosen as the base trajectory for the optimization. Cubic polynomials have been selected to approximate the SPCTOM because they are the lowest degree polynomials that result in a smooth curve, i.e., continuous and differentiable everywhere. The locations of the knots along the path have been chosen to be the same as the locations of the switching points of the PCTOM (Figure 2).

The variables of the optimization are the end-effector pseudo-velocities at the preselected knot-points along the path and the slopes of the trajectory in the s-$\dot{s}$ phase plane at the path end-points. Thus, the vector of optimization variables, $\mathbf{x}$, is defined as:

$$\mathbf{x} = \left(\frac{\left(\frac{d\dot{s}}{ds}\right)_0}{\left(\frac{d\dot{s}}{ds}\right)_{m,0}} \quad \frac{\dot{s}_1}{\dot{s}_{m,1}} \quad \cdots \quad \frac{\dot{s}_p}{\dot{s}_{m,p}} \quad \frac{\left(\frac{d\dot{s}}{ds}\right)_f}{\left(\frac{d\dot{s}}{ds}\right)_{m,f}} \right)^T, \quad (25)$$

where the values with the index m correspond to the limiting PCTOM (the dotted line in Figure 2), while the other values correspond to the splined trajectory (the solid line). These variables are normalized since the end slopes vary over a much wider range than the pseudo-velocities.

The optimal trajectory results from splining cubic polynomials in the s-$\dot{s}$ phase plane through the optimal $\mathbf{x}$ knots. The trajectory must be within actuator torque and actuator jerk limits and take minimum time. The control and

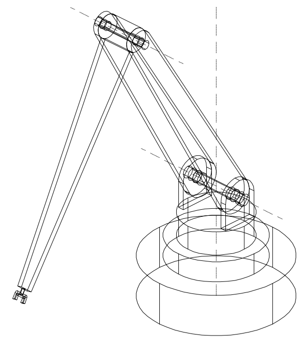


Figure 3: THE ELBOW MANIPULATOR.

state inequality constraints in Equations (20), (21), (22), and (23) thus become:

$$g_{4(i-1)+1}(\mathbf{x}) = 1 - \max_{\dot{s}(s)} \frac{T_i}{T_{max,i}}, \quad (26)$$

$$g_{4(i-1)+2}(\mathbf{x}) = 1 - \max_{\dot{s}(s)} \frac{T_i}{T_{min,i}}, \quad (27)$$

$$g_{4(i-1)+3}(\mathbf{x}) = 1 - \max_{\dot{s}(s)} \frac{\dot{T}_i}{\dot{T}_{max,i}}, \quad (28)$$

$$g_{4(i-1)+4}(\mathbf{x}) = 1 - \max_{\dot{s}(s)} \frac{\dot{T}_i}{\dot{T}_{min,i}}, \quad (29)$$

for $i = 1 \ldots n$. By this definition, whenever any of the actuator torques and/or jerks exceeds its limits, the respective constraint becomes negative.

As formulated, the optimization is solved herein using the flexible tolerance method (FTM) (Himmelblau, 1989)[4]. There are two reasons for choosing this method. First, the derivatives of the constraints and the cost function, i.e., motion time, are not available. Second, as described previously, the solution sought is expected to be on the boundary of the admissible region; therefore it is desirable to use information about points on both sides of the limiting surfaces in order to converge to the surface. The details of the FTM are discussed in the Appendix .

EXAMPLE

The method for determining optimal SPCTOM has been implemented in MATLAB and simulations are performed for the elbow manipulator presented schematically in Figure 3. The robot parameters are taken from (Pfeiffer, 1987)[9]. They are given in Table 1, where C.O.M. means the location of the center of mass of link i with respect to joint i.

Table 1: ROBOT PARAMETERS.

Link $[m]$	C.O.M. $[m]$	Mass $[kg]$
$l_1 = 0$	$l_{c1} = 0.05$	$m_1 = 0$
$l_2 = 0.75$	$l_{c2} = 0.2$	$m_2 = 6.6$
$l_3 = 0.75$	$l_{c3} = 0.15$	$m_3 = 4.2$
$I_x[kgm^2]$	$I_y[kgm^2]$	$I_z[kgm^2]$
$I_{x1} = 0$	$I_{y1} = 5$	$I_{z1} = 0$
$I_{x2} = 0.1$	$I_{y2} = .6$	$I_{z2} = 0.6$
$I_{x3} = 0.02$	$I_{y3} = .2$	$I_{z3} = 0.3$

Table 2: ACTUATOR PARAMETERS.

$T[Nm]$	$\dot{T}_1[Nm/s]$	$\dot{T}_2[Nm/s]$	$\dot{T}_3[Nm/s]$
$T_1 = 140$	$\dot{T}_{11} = 5000$	$\dot{T}_{12} = 500$	$\dot{T}_{13} = 140$
$T_2 = 140$	$\dot{T}_{21} = 2000$	$\dot{T}_{22} = 500$	$\dot{T}_{23} = 140$
$T_3 = 50$	$\dot{T}_{31} = 1000$	$\dot{T}_{32} = 100$	$\dot{T}_{33} = 50$

The actuator torque limits are the same for all the three examples given in this paper, while the limits on the jerks are different, as successively shown in Table 2.

The resulting optimal trajectories for the different limits on the actuator jerks are shown in Figures 4(a), 5(a) and 6(a), respectively, by solid lines. The dashed lines represent the time-optimal trajectory considering only torque limits (PCTOM). The dotted lines are the smooth motion velocity limit curves (SMVLC), i.e., the velocity limit curves determined considering both torque and jerk limits. The corresponding actuator torques and jerks are also plotted in these figures.

While the PCTOM takes 1.72 seconds, the SPCTOM takes 1.91 seconds in the first example. Here, the limits on the actuator jerks were very high and the trajectory is determined by the limits on the actuator torques. Although one would expect both trajectories to yield same motion times, there are two reasons for the increase in motion time for SPCTOM : (i) the limited parameterization chosen in the $s - \dot{s}$ phase plane and (ii) the significant decrease in peak actuator jerks for SPCTOM (solid lines) compared to PCTOM (dotted lines), as shown in Figure 7.

In examples 2 and 3, the limits on the actuator jerks predominate, therefore the torque constraints are not approached. The optimal motion times for these examples are significantly higher, 2.74 seconds and 3.93 seconds, respectively.

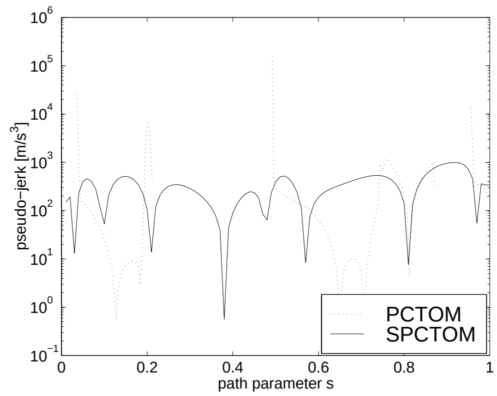


Figure 7: ACTUATOR JERKS FOR THE SPCTOM IN EXAMPLE 1 (SOLID LINES) AND PCTOM (DOTTED LINES).

DISCUSSION

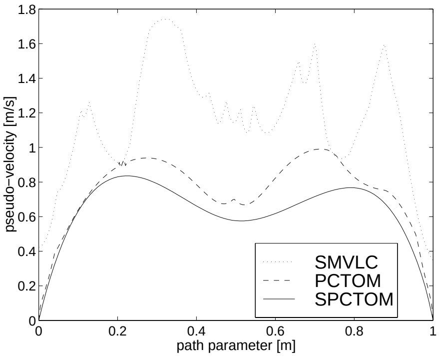
As desired, the optimal trajectories determined through the proposed method are not bang-bang in the controls. This is a consequence of the parameterization in the phase plane, where the trajectory is approximated by splining cubic polynomials. However, as seen from the first example presented, the chosen parameterization by itself does not cause a significant increase in the motion time.

The limits on the actuator jerks, on the other hand, affect the motion time substantially. As expected, the more restrictive the limits on actuator jerks are, the higher the motion time is. Therefore, these limits should be chosen based on the characteristics of the joint actuators, such that a suitable compromise is achieved between the motion time and the life span of the actuators.

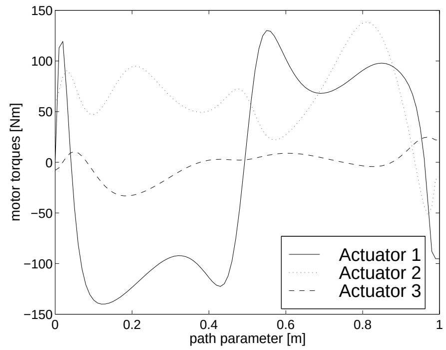
CONCLUSIONS

A method has been presented for determining smooth and time-optimal path-constrained trajectories for robotic manipulators. The dynamics of the manipulator together with limits on the actuator torques and jerks are considered. The trajectory in the $s - \dot{s}$ phase plane is parameterized by cubic splines and the optimal motion is obtained through an optimization of a base trajectory.

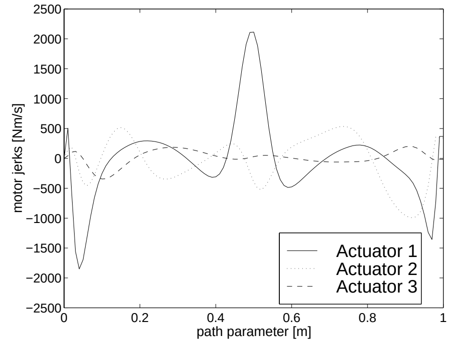
The limits on the actuator jerks are a simple and direct way of adjusting the smoothness of the end-effector motion or of each link individually. More than that, they are actuator dependent and position independent. Thus, they represent a uniform measure for the trajectory smoothness over the entire robot workspace. This results in faster tra-



(a) SPCTOM TRAJECTORY IN THE $s - \dot{s}$ PLANE.

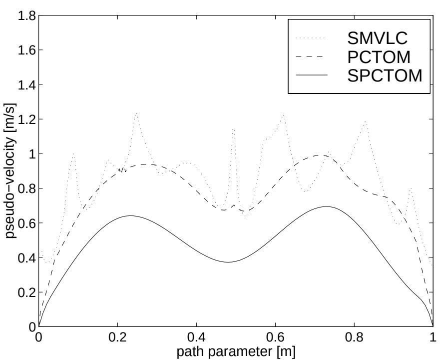


(b) ACTUATOR TORQUES.

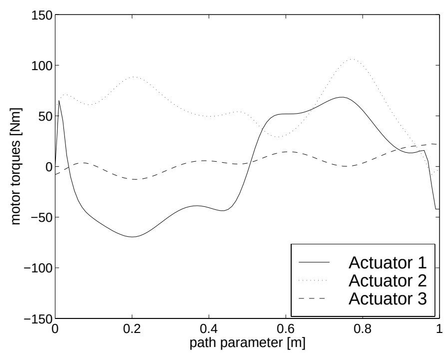


(c) ACTUATOR JERKS.

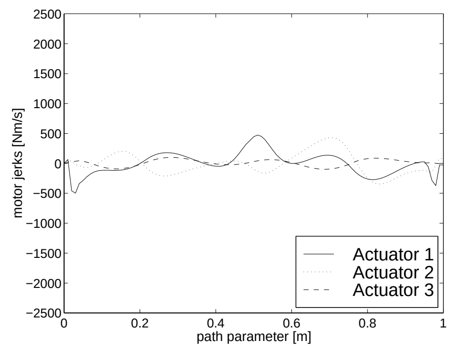
Figure 4: EXAMPLE 1 (HIGH JERK LIMITS).



(a) SPCTOM TRAJECTORY IN THE $s - \dot{s}$ PLANE.

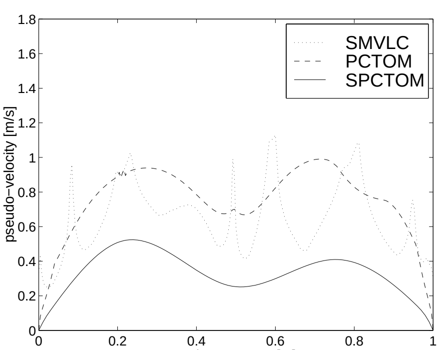


(b) ACTUATOR TORQUES.

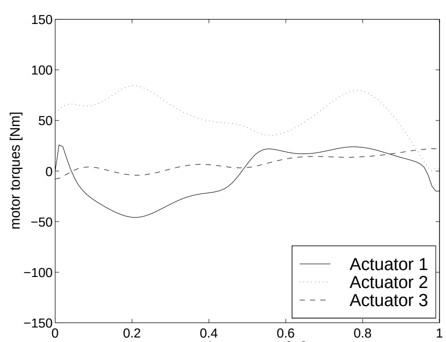


(c) ACTUATOR JERKS.

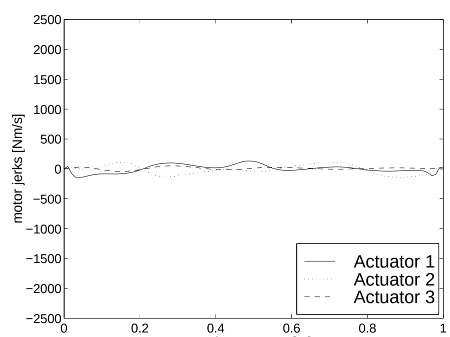
Figure 5: EXAMPLE 2 (MEDIUM JERK LIMITS).



(a) SPCTOM TRAJECTORY IN THE $s - \dot{s}$ PLANE.



(b) ACTUATOR TORQUES.



(c) ACTUATOR JERKS.

Figure 6: EXAMPLE 3 (LOW JERK LIMITS).

jectories compared to other smooth time-optimal trajectories that use global velocity and acceleration limits.

Smooth time-optimal trajectories planned according to the method proposed herein maintain a measure of time optimality while achieving a desired degree of smoothness. Therefore, they are suitable for direct implementation on a light direct drive manipulator, since they do not excite the eigenfrequencies of the robot structure. Experimentally, they have been shown to compensate for large modelling errors.

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Appendix

In the flexible tolerance method (FTM) (Himmelblau, 1989)[4], the optimization problem:

$$\text{Minimize:} \quad f(\mathbf{x}) \quad \mathbf{x} \in \mathbf{R}^n \tag{.1}$$

$$\text{Subject to:} \quad h_i(\mathbf{x}) = 0 \quad i = 1, ..., m \tag{.2}$$

$$g_i(\mathbf{x}) \geq 0 \quad i = m+1, ..., p$$

is solved as the following simpler equivalent problem with only one constraint:

$$\text{min:} \quad f(\mathbf{x}) \quad \mathbf{x} \in \mathbf{R}^n \tag{.3}$$

$$\text{subject to:} \quad \Phi^{(k)} - \mathcal{T}(\mathbf{x}) \geq \prime.$$

$\Phi^{(k)}$ is the value of the flexible tolerance criterion at the kth step of the optimization and it also serves as a criterion for the termination of the search. The cost function $f(\mathbf{x})$ and the equality and inequality constraints in (.3) may be linear and/or non-linear functions of the variables in $\mathbf{x}$. The value of the cost function is improved by using information provided by feasible points, as well as certain nonfeasible points called *near-feasible points*. The near-feasibility limits are made more restrictive as the search advances, until in the limit only feasible points are accepted.

In (.4) below, $\mathcal{T}(\mathbf{x})$ is a positive functional of all the equality and/or inequality constraints of the original problem and it is used as a measure of the constraint violation, while Φ is selected as a positive decreasing function of the $\mathbf{x}$ points in $\mathbf{R}^n$. For the SPCTOM:

$$\mathcal{T}(\mathbf{x}) = \begin{cases} \max_i g_i(\mathbf{x}) & \text{if} \quad \exists_i \quad \text{such that} \quad g_i(\mathbf{x}) \geq 1 \\ 0 & \text{otherwise,} \end{cases} \qquad (.4)$$

and:

$$\Phi^{(k)} = \min\{\Phi^{(k-1)}; \kappa \sum_{i=1}^{r+1} \|x_i^{(k)} - x_{centr}^{(k)}\|\} \qquad (.5)$$

with κ a constant.

The tolerance criterion is used to classify points in $\mathbf{R}^n$. At the kth step of the optimization, a point $\mathbf{x}^{(k)}$ is said to be:

1. Feasible, if $\mathcal{T}(\mathbf{x}) = /$
2. Near-feasible, if $0 \leq \mathcal{T}(\mathbf{x}) \leq \Phi^{(\|)}$
3. Nonfeasible, if $\mathcal{T}(\mathbf{x}) < \Phi^{(\|)}$.

A small value of $\mathcal{T}(\mathbf{x}^{(\|)})$ implies that $\mathbf{x}^{(k)}$ is relatively near to the feasible region, and a large value of $\mathcal{T}(\mathbf{x}^{(\|)})$ implies that $\mathbf{x}^{(k)}$ is relatively far from the feasible region.

On a transition from $\mathbf{x}^{(k)}$ to $\mathbf{x}^{(k+1)}$, the move is said to be feasible if $0 \leq \mathcal{T}(\mathbf{x}^{(\|+\infty)}) \leq \Phi^{(\|)}$, and nonfeasible if $\Phi^{(k)} \leq \mathcal{T}(\mathbf{x}^{(\|+\infty)})$.

The FTM entails two independent optimizations : an outer minimization of the cost function $f(\mathbf{x})$ and an inner minimization of the violation of constraints $\mathcal{T}(\mathbf{x})$ whenever the minimization of $f(\mathbf{x})$ yields an infeasible point. The outer optimization of the motion time is implemented in this paper using the flexible polyhedron method (FPM) (Nelder, 1964)[8]. The FPM is a search in n dimensions where the polyhedron changes shape to match the changing shape of the surface. In the vicinity of a minimum the polyhedron shrinks, surrounding the minimum. Replacement of an infeasible point with a feasible or near feasible one is done through a line search using interval halving.