

B-Spline Example 1

Expand the equation of a nonperiodic uniform B-spline curve of order 3 in polynomial form. Assume that the control points of the curve are \mathbf{P}_0 , \mathbf{P}_1 , and \mathbf{P}_2 .

ANSWER

From Equation (6.35), the knot values t_i are

$$t_0 = 0, \quad t_1 = 0, \quad t_2 = 0, \quad t_3 = 1, \quad t_4 = 1, \quad t_5 = 1$$

and the parameter u ranges from 0 to 1. Let's use Equation (6.33) to obtain the blending functions of order 1, $N_{i,1}(u)$:

$$N_{0,1}(u) = \begin{cases} 1 & t_0 \leq u \leq t_1 \quad (u = 0) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{1,1}(u) = \begin{cases} 1 & t_1 \leq u \leq t_2 \quad (u = 0) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{2,1}(u) = \begin{cases} 1 & t_2 \leq u \leq t_3 \quad (u \leq 1) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{3,1}(u) = \begin{cases} 1 & t_3 \leq u \leq t_4 \quad (u = 1) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{4,1}(u) = \begin{cases} 1 & t_4 \leq u \leq t_5 \quad (u = 1) \\ 0 & \text{otherwise} \end{cases}$$

From among $N_{0,1}(u)$, $N_{1,1}(u)$, and $N_{2,1}(u)$, we choose $N_{2,1}(u)$ to be the blending function having nonzero value at $u = 0$. Similarly, we choose $N_{2,1}(u)$ from among $N_{2,1}(u)$, $N_{3,1}(u)$, and $N_{4,1}(u)$ to be the blending function having nonzero value at $u = 1$. Thus $N_{2,1}(u)$ becomes the only nonzero blending function of order 1 in the parameter range $[0,1]$ and has a constant value of 1 over the entire range.

Thus we obtain the nontrivial blending functions of order 2 from Equation (6.32) as follows:¹³

$$\begin{aligned}
 N_{1,2}(u) &= \frac{(u-t_1)N_{1,1}}{t_2-t_1} + \frac{(t_3-u)N_{2,1}}{t_3-t_2} = \frac{(1-u)N_{2,1}}{1} \\
 &= (1-u) \\
 N_{2,2}(u) &= \frac{(u-t_2)N_{2,1}}{t_3-t_2} + \frac{(t_4-u)N_{3,1}}{t_4-t_3} = \frac{uN_{2,1}}{1} \\
 &= u
 \end{aligned}$$

Similarly, we get the blending functions of order 3, $N_{i,3}(u)$:

$$\begin{aligned}
 N_{0,3}(u) &= \frac{(u-t_0)N_{0,2}}{t_3-t_1} + \frac{(t_3-u)N_{1,2}}{t_4-t_2} = \frac{(1-u)N_{1,2}}{1} = (1-u)^2 \\
 N_{1,3}(u) &= \frac{(u-t_1)N_{1,2}}{t_3-t_1} + \frac{(t_4-u)N_{2,2}}{t_4-t_2} = u(1-u) + (1-u)u = 2u(1-u) \\
 N_{2,3}(u) &= \frac{(u-t_2)N_{2,2}}{t_4-t_2} + \frac{(t_5-u)N_{3,2}}{t_5-t_3} \\
 &= u^2
 \end{aligned}$$

Then the expanded equation of the B-spline curve is

$$\mathbf{P}(u) = (1-u)^2\mathbf{P}_0 + 2u(1-u)\mathbf{P}_1 + u^2\mathbf{P}_2 \quad (6.36)$$

The equation of the Bezier curve defined by the control points \mathbf{P}_0 , \mathbf{P}_1 , and \mathbf{P}_2 can be expanded as

$$\begin{aligned}
 \mathbf{P}(u) &= \binom{2}{0} u^0(1-u)^2\mathbf{P}_0 + \binom{2}{1} u^1(1-u)^1\mathbf{P}_1 + \binom{2}{2} u^2(1-u)^0\mathbf{P}_2 \\
 &= (1-u)^2\mathbf{P}_0 + 2u(1-u)\mathbf{P}_1 + u^2\mathbf{P}_2
 \end{aligned} \quad (6.37)$$

Comparing Equations (6.36) and (6.37) shows that the nonperiodic, uniform B-spline curve of order 3 defined by the control points \mathbf{P}_0 , \mathbf{P}_1 , and \mathbf{P}_2 is the same as the Bezier curve defined by the same control points. Thus, in general, we can say that a nonperiodic, uniform B-spline curve is the same as the Bezier curve defined by the same control points if order k equals the number of the control points ($n + 1$). In other words, a Bezier curve is simply a special case of a B-spline curve.

B-Spline Example 1

Expand the equation of a nonperiodic, uniform B-spline curve of order 3 defined by the control points $\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_5$ in polynomial form and show its local modification capability.

ANSWER

From Equation (6.35), the knot values t_i are

$$t_0 = 0, \quad t_1 = 0, \quad t_2 = 0, \quad t_3 = 1, \quad t_4 = 2, \quad t_5 = 3, \quad t_6 = 4, \quad t_7 = 4, \quad t_8 = 4$$

and the parameter u will range from 0 to 4. Let's use Equation (6.33) to obtain the blending functions of order 1, $N_{i,1}(u)$:

$$N_{2,1}(u) = \begin{cases} 1 & 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{3,1}(u) = \begin{cases} 1 & 1 \leq u \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{4,1}(u) = \begin{cases} 1 & 2 \leq u \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{5,1}(u) = \begin{cases} 1 & 3 \leq u \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

We ignore $N_{0,1}(u)$ and $N_{1,1}(u)$ by choosing $N_{2,1}(u)$ to be the only nonzero blending function at $u = 0$. Similarly, we ignore $N_{6,1}(u)$ and $N_{7,1}(u)$. Now we obtain the non-trivial blending functions of order 2 from Equation (6.32) as follows:

$$N_{1,2}(u) = \frac{(u-t_1)N_{1,1}}{t_2-t_1} + \frac{(t_3-u)N_{2,1}}{t_3-t_2} = (1-u)N_{2,1}$$

$$N_{2,2}(u) = \frac{(u-t_2)N_{2,1}}{t_3-t_2} + \frac{(t_4-u)N_{3,1}}{t_4-t_3} = uN_{2,1} + (2-u)N_{3,1}$$

$$N_{3,2}(u) = \frac{(u-t_3)N_{3,1}}{t_4-t_3} + \frac{(t_5-u)N_{4,1}}{t_5-t_4} = (u-1)N_{3,1} + (3-u)N_{4,1}$$

$$N_{4,2}(u) = \frac{(u-t_4)N_{4,1}}{t_5-t_4} + \frac{(t_6-u)N_{5,1}}{t_6-t_5} = (u-2)N_{4,1} + (4-u)N_{5,1}$$

$$N_{5,2}(u) = \frac{(u-t_5)N_{5,1}}{t_6-t_5} + \frac{(t_7-u)N_{6,1}}{t_7-t_6} = (u-3)N_{5,1}$$

Similarly, we get the blending functions of order 3, $N_{i,3}(u)$:

$$N_{0,3}(u) = \frac{(u-t_0)N_{0,2}}{t_2-t_0} + \frac{(t_3-u)N_{1,2}}{t_3-t_1} = (1-u)N_{1,2} = (1-u)^2 N_{2,1}$$

$$N_{1,3}(u) = \frac{(u-t_1)N_{1,2}}{t_3-t_1} + \frac{(t_4-u)N_{2,2}}{t_4-t_2} = uN_{1,2} + \frac{2-u}{2} N_{2,2}$$

$$= \left[u(1-u) + \frac{(2-u)u}{2} \right] N_{2,1} + \frac{(2-u)^2}{2} N_{3,1}$$

$$N_{2,3}(u) = \frac{(u-t_2)N_{2,2}}{t_4-t_2} + \frac{(t_5-u)N_{3,2}}{t_5-t_3} = \frac{u}{2} N_{2,2} + \frac{3-u}{2} N_{3,2}$$

$$= \frac{u^2}{2} N_{2,1} + \left[\frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2} \right] N_{3,1} + \frac{(3-u)^2}{2} N_{4,1}$$

$$\begin{aligned}
N_{3,3}(u) &= \frac{(u-t_3)N_{3,2}}{t_5-t_3} + \frac{(t_6-u)N_{4,2}}{t_6-t_4} = \frac{u-1}{2}N_{3,2} + \frac{4-u}{2}N_{4,2} \\
&= \frac{(u-1)^2}{2}N_{3,1} + \left[\frac{(u-1)(3-u)}{2} + \frac{(4-u)(u-2)}{2} \right]N_{4,1} + \frac{(4-u)^2}{2}N_{5,1}
\end{aligned}$$

$$\begin{aligned}
N_{4,3}(u) &= \frac{(u-t_4)N_{4,2}}{t_6-t_4} + \frac{(t_7-u)N_{5,2}}{t_7-t_5} = \frac{u-2}{2}N_{4,2} + (4-u)N_{5,2} \\
&= \frac{(u-2)^2}{2}N_{4,1} + \left[\frac{(u-2)(4-u)}{2} + (4-u)(u-3) \right]N_{5,1}
\end{aligned}$$

$$N_{5,3}(u) = \frac{(u-t_5)N_{5,2}}{t_7-t_5} + \frac{(t_8-u)N_{6,2}}{t_8-t_6} = (u-3)N_{5,2} = (u-3)^2N_{5,1}$$

Then the expanded equation of the B-spline curve is

$$\begin{aligned}
 \mathbf{P}(u) = & (1-u)^2 N_{2,1} \mathbf{P}_0 + \left\{ \left[u(1-u) + \frac{(2-u)u}{2} \right] N_{2,1} + \frac{(2-u)^2}{2} N_{3,1} \right\} \mathbf{P}_1 \\
 & + \left\{ \frac{u^2}{2} N_{2,1} + \left[\frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2} \right] N_{3,1} + \frac{(3-u)^2}{2} N_{4,1} \right\} \mathbf{P}_2 \\
 & + \left\{ \frac{(u-1)^2}{2} N_{3,1} + \left[\frac{(u-1)(3-u)}{2} + \frac{(4-u)(u-2)}{2} \right] N_{4,1} + \frac{(4-u)^2}{2} N_{5,1} \right\} \mathbf{P}_3 \quad (6.38) \\
 & + \left\{ \frac{(u-2)^2}{2} N_{4,1} + \left[\frac{(u-2)(4-u)}{2} + (4-u)(u-3) \right] N_{5,1} \right\} \mathbf{P}_4 \\
 & + (u-3)^2 N_{5,1} \mathbf{P}_5
 \end{aligned}$$