B-Spline Example 1

Expand the equation of a nonperiodic uniform B-spline curve of order 3 in polynomial form. Assume that the control points of the curve are $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$.

ANSWER

From Equation (6.35), the knot values t_i are

$$t_0 = 0, \quad t_1 = 0, \quad t_2 = 0, \quad t_3 = 1, \quad t_4 = 1, \quad t_5 = 1$$

and the parameter u ranges from 0 to 1. Let's use Equation (6.33) to obtain the blending functions of order 1, $N_{i,1}(u)$:

$$N_{0,1}(u) = \begin{cases} 1 & t_0 \le u \le t_1 \quad (u = 0) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{1,1}(u) = \begin{cases} 1 & t_1 \le u \le t_2 \quad (u = 0) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{2,1}(u) = \begin{cases} 1 & t_2 \le u \le t_3 \quad (u \le 1) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{3,1}(u) = \begin{cases} 1 & t_3 \le u \le t_4 \quad (u = 1) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{4,1}(u) = \begin{cases} 1 & t_4 \le u \le t_5 \quad (u = 1) \\ 0 & \text{otherwise} \end{cases}$$

From among $N_{0,1}(u)$, $N_{1,1}(u)$, and $N_{2,1}(u)$, we choose $N_{2,1}(u)$ to be the blending function having nonzero value at $u = 0$. Similarly, we choose $N_{2,1}(u)$ from among $N_{2,1}(u)$, $N_{3,1}(u)$, and $N_{4,1}(u)$ to be the blending function having nonzero value at $u = 1$. Thus $N_{2,1}(u)$ becomes the only nonzero blending function of order 1 in the parameter range [0,1] and has a constant value of 1 over the entire range.

Thus we obtain the nontrivial blending functions of order 2 from Equation (6.32) as follows:[13]

$$
\begin{aligned}
N_{1,2}(u) &= \frac{(u-t_1)N_{1,1}}{t_2-t_1} + \frac{(t_3-u)N_{2,1}}{t_3-t_2} = \frac{(1-u)N_{2,1}}{1} \\
&= (1-u) \\
N_{2,2}(u) &= \frac{(u-t_2)N_{2,1}}{t_3-t_2} + \frac{(t_4-u)N_{3,1}}{t_4-t_3} = \frac{uN_{2,1}}{1} \\
&= u
\end{aligned}
$$

Similarly, we get the blending functions of order 3, $N_{i,3}(u)$:

$$N_{0,3}(u) = \frac{(u - t_0)N_{0,2}}{t_3 - t_1} + \frac{(t_3 - u)N_{1,2}}{t_4 - t_2} = \frac{(1-u)N_{1,2}}{1} = (1-u)^2$$

$$N_{1,3}(u) = \frac{(u - t_1)N_{1,2}}{t_3 - t_1} + \frac{(t_4 - u)N_{2,2}}{t_4 - t_2} = u(1-u) + (1-u)u = 2u(1-u)$$

$$N_{2,3}(u) = \frac{(u - t_2)N_{2,2}}{t_4 - t_2} + \frac{(t_5 - u)N_{3,2}}{t_5 - t_3}$$
$$= u^2$$

Then the expanded equation of the B-spline curve is

$$\mathbf{P}(u) = (1-u)^2\mathbf{P}_0 + 2u\,(1-u)\mathbf{P}_1 + u^2\mathbf{P}_2 \tag{6.36}$$

The equation of the Bezier curve defined by the control points $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$ can be expanded as

$$\begin{aligned}\mathbf{P}(u) &= \binom{2}{0}_0 u^0(1-u)^2\mathbf{P}_0 + \binom{2}{1}_1 u^1(1-u)^1\mathbf{P}_1 + \binom{2}{2}_2 u^2(1-u)^0\mathbf{P}_2 \\ &= (1-u)^2\mathbf{P}_0 + 2u(1-u)\mathbf{P}_1 + u^2\mathbf{P}_2\end{aligned} \tag{6.37}$$

Comparing Equations (6.36) and (6.37) shows that the nonperiodic, uniform B-spline curve of order 3 defined by the control points $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$ is the same as the Bezier curve defined by the same control points. Thus, in general, we can say that a nonperiodic, uniform B-spline curve is the same as the Bezier curve defined by the same control points if order k equals the number of the control points $(n + 1)$. In other words, a Bezier curve is simply a special case of a B-spline curve.

B-Spline Example 1

Expand the equation of a nonperiodic, uniform B-spline curve of order 3 defined by the control points $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_5$ in polynomial form and show its local modification capability.

ANSWER

From Equation (6.35), the knot values t_1 are

$$t_0 = 0, \quad t_1 = 0, \quad t_2 = 0, \quad t_3 = 1, \quad t_4 = 2, \quad t_5 = 3, \quad t_6 = 4, \quad t_7 = 4, \quad t_8 = 4$$

and the parameter u will range from 0 to 4. Let's use Equation (6.33) to obtain the blending functions of order 1, $N_{i,1}(\mathrm{u})$:

$$N_{2,1}(u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{3,1}(u) = \begin{cases} 1 & 1 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{4,1}(u) = \begin{cases} 1 & 2 \le u \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{5,1}(u) = \begin{cases} 1 & 3 \le u \le 4 \\ 0 & \text{otherwise} \end{cases}$$

We ignore $N_{0,1}(u)$ and $N_{1,1}(u)$ by choosing $N_{2,1}(u)$ to be the only nonzero blending function at $u = 0$. Similarly, we ignore $N_{6,1}(u)$ and $N_{7,1}(u)$. Now we obtain the non-trivial blending functions of order 2 from Equation (6.32) as follows:

$$N_{1,2}(u) = \frac{(u-t_1)N_{1,1}}{t_2-t_1} + \frac{(t_3-u)N_{2,1}}{t_3-t_2} = (1-u)N_{2,1}$$

$$N_{2,2}(u) = \frac{(u-t_2)N_{2,1}}{t_3-t_2} + \frac{(t_4-u)N_{3,1}}{t_4-t_3} = uN_{2,1} + (2-u)N_{3,1}$$

$$N_{3,2}(u) = \frac{(u-t_3)N_{3,1}}{t_4-t_3} + \frac{(t_5-u)N_{4,1}}{t_5-t_4} = (u-1)N_{3,1} + (3-u)N_{4,1}$$

$$N_{4,2}(u) = \frac{(u-t_4)N_{4,1}}{t_5-t_4} + \frac{(t_6-u)N_{5,1}}{t_6-t_5} = (u-2)N_{4,1} + (4-u)N_{5,1}$$

$$N_{5,2}(u) = \frac{(u-t_5)N_{5,1}}{t_6-t_5} + \frac{(t_7-u)N_{6,1}}{t_7-t_6} = (u-3)N_{5,1}$$

Similarly, we get the blending functions of order 3, $N_{i,3}(u)$:

$$N_{0,3}(u) = \frac{(u-t_0)N_{0,2}}{t_2-t_0} + \frac{(t_3-u)N_{1,2}}{t_3-t_1} = (1-u)N_{1,2} = (1-u)^2 N_{2,1}$$

$$N_{1,3}(u) = \frac{(u-t_1)N_{1,2}}{t_3-t_1} + \frac{(t_4-u)N_{2,2}}{t_4-t_2} = uN_{1,2} + \frac{2-u}{2}N_{2,2}$$

$$= \left[u(1-u) + \frac{(2-u)u}{2}\right]N_{2,1} + \frac{(2-u)^2}{2}N_{3,1}$$

$$N_{2,3}(u) = \frac{(u-t_2)N_{2,2}}{t_4-t_2} + \frac{(t_5-u)N_{3,2}}{t_5-t_3} = \frac{u}{2}N_{2,2} + \frac{3-u}{2}N_{3,2}$$

$$= \frac{u^2}{2}N_{2,1} + \left[\frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2}\right]N_{3,1} + \frac{(3-u)^2}{2}N_{4,1}$$

$$N_{3,3}(u) = \frac{(u-t_3)N_{3,2}}{t_5-t_3} + \frac{(t_6-u)N_{4,2}}{t_6-t_4} = \frac{u-1}{2}N_{3,2} + \frac{4-u}{2}N_{4,2}$$

$$= \frac{(u-1)^2}{2}N_{3,1} + \left[\frac{(u-1)(3-u)}{2} + \frac{(4-u)(u-2)}{2}\right]N_{4,1} + \frac{(4-u)^2}{2}N_{5,1}$$

$$N_{4,3}(u) = \frac{(u-t_4)N_{4,2}}{t_6-t_4} + \frac{(t_7-u)N_{5,2}}{t_7-t_5} = \frac{u-2}{2}N_{4,2} + (4-u)N_{5,2}$$

$$= \frac{(u-2)^2}{2}N_{4,1} + \left[\frac{(u-2)(4-u)}{2} + (4-u)(u-3)\right]N_{5,1}$$

$$N_{5,3}(u) = \frac{(u-t_5)N_{5,2}}{t_7-t_5} + \frac{(t_8-u)N_{6,2}}{t_8-t_6} = (u-3)N_{5,2} = (u-3)^2 N_{5,1}$$

Then the expanded equation of the B-spline curve is

$$
\begin{aligned}
\mathbf{P}(u) = {} & (1-u)^2 N_{2,1}\mathbf{P}_0 + \left\{\left[u(1-u)+\frac{(2-u)u}{2}\right]N_{2,1} + \frac{(2-u)^2}{2}N_{3,1}\right\}\mathbf{P}_1 \\
& + \left\{\frac{u^2}{2}N_{2,1} + \left[\frac{u(2-u)}{2}+\frac{(3-u)(u-1)}{2}\right]N_{3,1} + \frac{(3-u)^2}{2}N_{4,1}\right\}\mathbf{P}_2 \\
& + \left\{\frac{(u-1)^2}{2}N_{3,1} + \left[\frac{(u-1)(3-u)}{2}+\frac{(4-u)(u-2)}{2}\right]N_{4,1} + \frac{(4-u)^2}{2}N_{5,1}\right\}\mathbf{P}_3 \\
& + \left\{\frac{(u-2)^2}{2}N_{4,1} + \left[\frac{(u-2)(4-u)}{2}+(4-u)(u-3)\right]N_{5,1}\right\}\mathbf{P}_4 \\
& + (u-3)^2 N_{5,1}\mathbf{P}_5
\end{aligned}
\tag{6.38}
$$