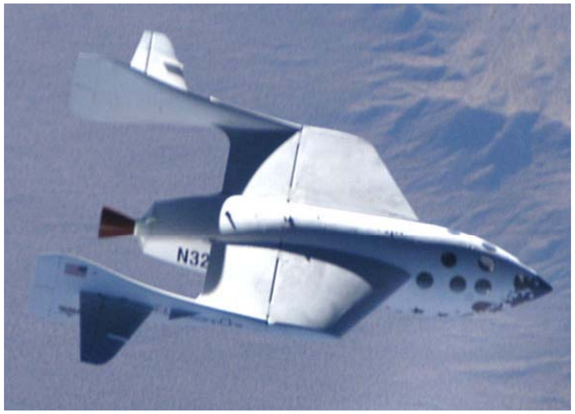
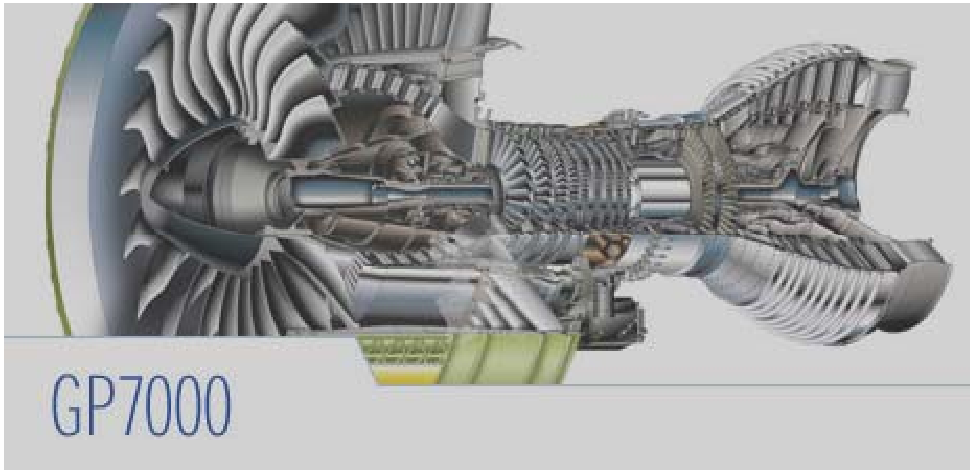
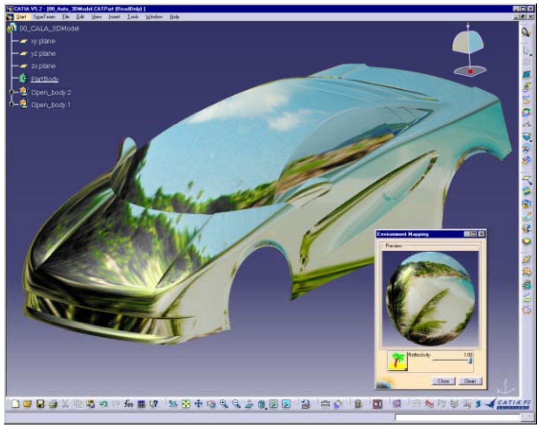
Free-form Surface I







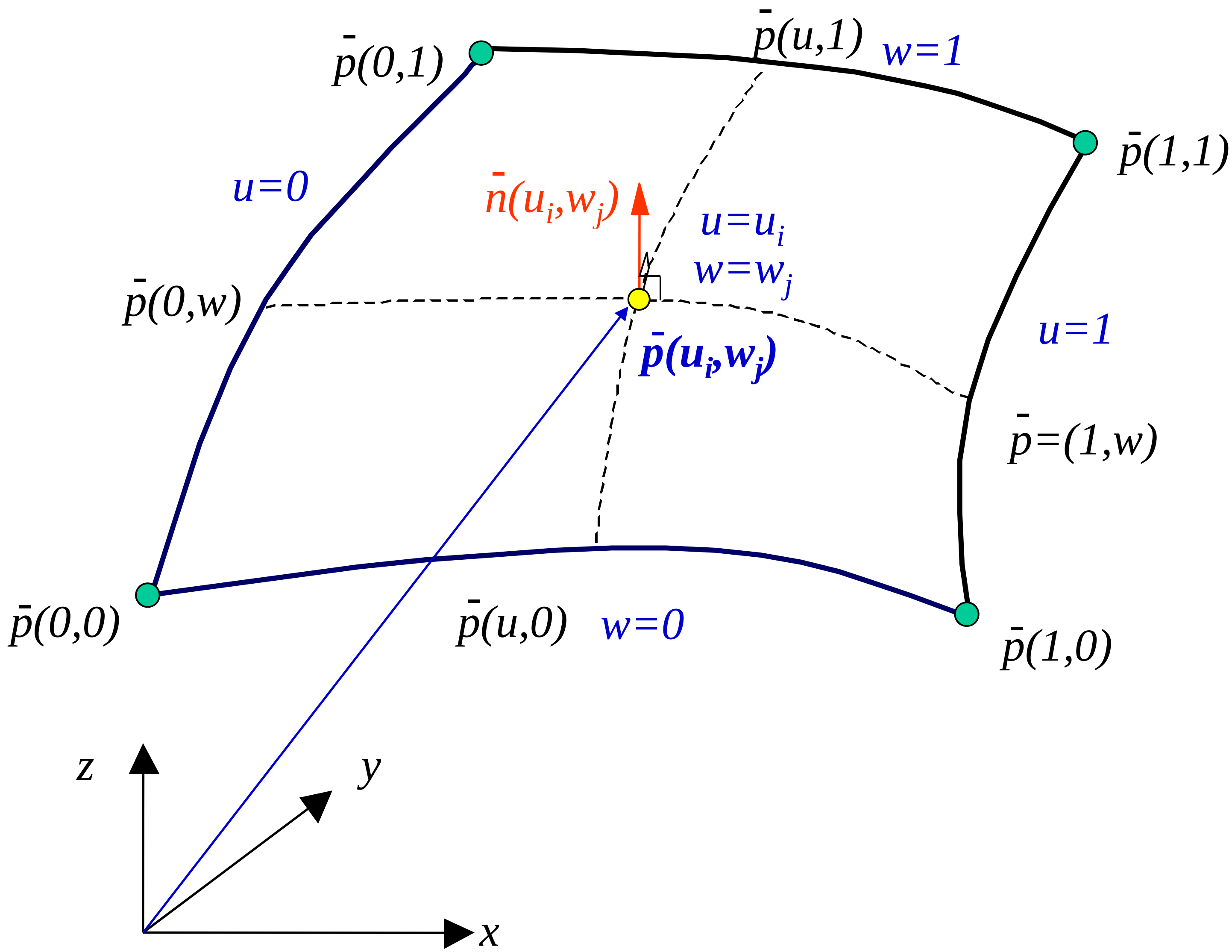
Applications of Complex Surfaces

Surface Patch

A surface patch — a curved bounded collection of points whose coordinates are given by continuous, two-parameter, single-valued mathematical expression.

Function of the form:

$$\overline{p}(u,w) = [x(u,w) \quad y(u,w) \quad z(u,w)]^T$$



p̄(0,1)
p̄(u,1)
w=1
p̄(1,1)
u=0
n̄(u_i,w_j)
u=u_i
w=w_j
p̄(0,w)
p̄(u_i,w_j)
u=1
p̄=(1,w)
p̄(0,0)
p̄(u,0)
w=0
p̄(1,0)
z
y
x

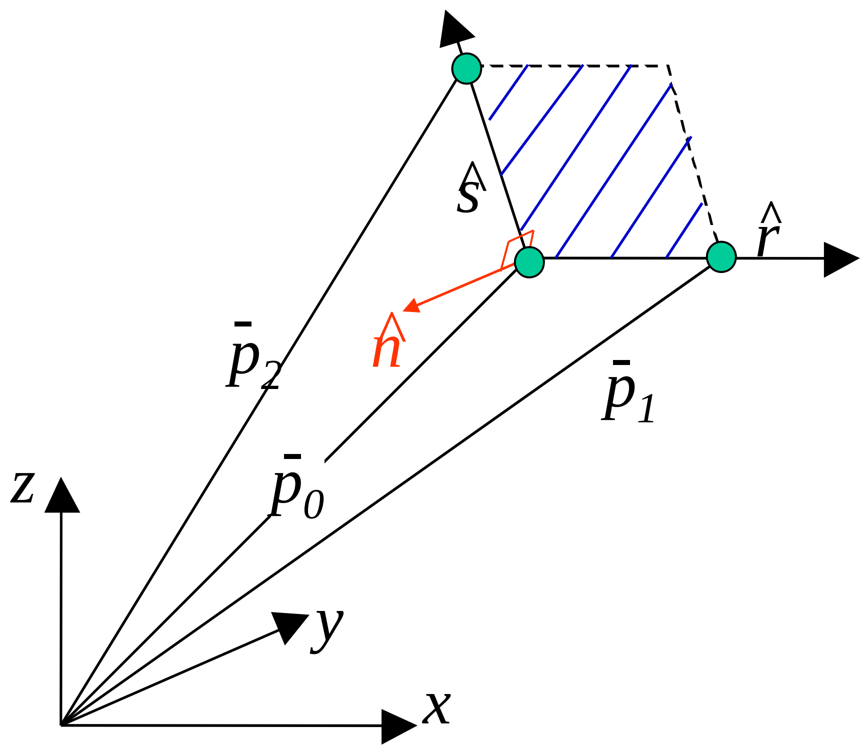
Type of Surface

- Planar Surface
- Bilinear Surface
- Ruled (Lofted) Surface
- Bi-cubic Surface
- Bezier Surface
- B-spline Surface

Planar Surface

- defined by three points and vectors

$$\overline{p}(u,w) = \overline{p_0} + u\left(\overline{p_1} - \overline{p_0}\right) + w\left(\overline{p_2} - \overline{p_0}\right) \qquad 0 \le u \le 1; \quad 0 \le w \le 1$$



Planar Surface

$$\overline{p}(u,w) = \overline{p_0} + u\left(\overline{p_1} - \overline{p_0}\right) + w\left(\overline{p_2} - \overline{p_0}\right) \qquad 0 \le u \le 1; \quad 0 \le w \le 1$$

$$\overline{p}(u,w) = \overline{p_0} + u\left|\overline{p_1} - \overline{p_0}\right|\hat{r} + w\left|\overline{p_2} - \overline{p_0}\right|\hat{s} \qquad 0 \le u \le 1; \quad 0 \le w \le 1$$

$$\hat{n} = \hat{r} \times \hat{s}$$ — surface normal

$$\hat{r} = \frac{\overline{p_1} - \overline{p_0}}{\left|\overline{p_1} - \overline{p_0}\right|}; \quad \hat{s} = \frac{\overline{p_2} - \overline{p_0}}{\left|\overline{p_2} - \overline{p_0}\right|}$$

Normalized Direction Vectors

Planar Surface

$$\overline{p}(u,w) = \overline{p_0} + u\left|\overline{p_1} - \overline{p_0}\right|\hat{r} + w\left|\overline{p_2} - \overline{p_0}\right|\hat{s} \qquad 0 \le u \le 1; \quad 0 \le w \le 1$$

$$Ax + By + Cz + D = 0$$

$$\hat{n} = A\hat{i} + B\hat{j} + C\hat{k}$$

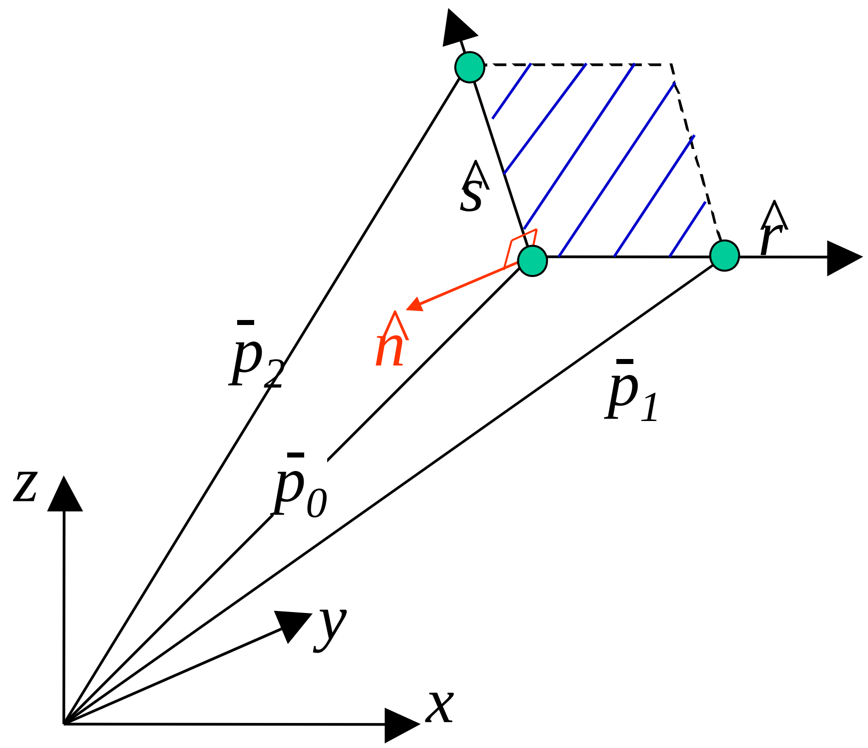
$$\hat{n} = \hat{r} \times \hat{s}$$

$$(\overline{P} - \overline{P_0}) \cdot \hat{n} = 0$$

$$(x - x_0) * A + (y - y_0) * B + (z - z_0) * C = 0$$

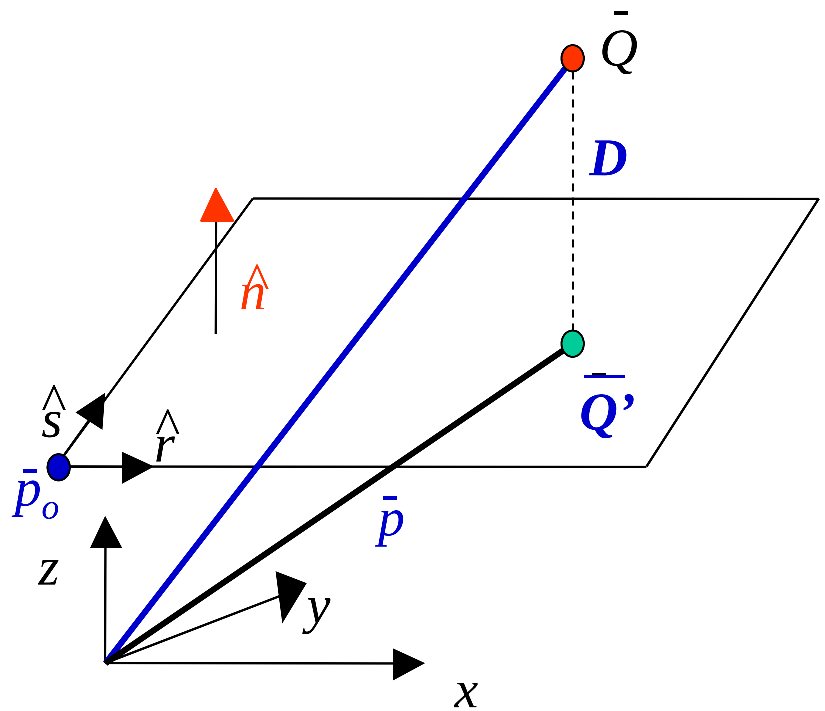
If define $D = -(Ax_0 + By_0 + Cz_0)$, then

$$Ax + By + Cz + D = 0$$



An Example

Find the distance between a point $\bar{Q} = [x_q \quad y_q \quad z_q]^T$ and a plane $\bar{p} = \bar{p}_0 + u\hat{r} + w\hat{s} \quad (0 \le u \le 1,\ 0 \le w \le 1)$. That is to say, find the projection of point $\bar{Q}$ onto plane $\bar{P}$ and the distance *D*.

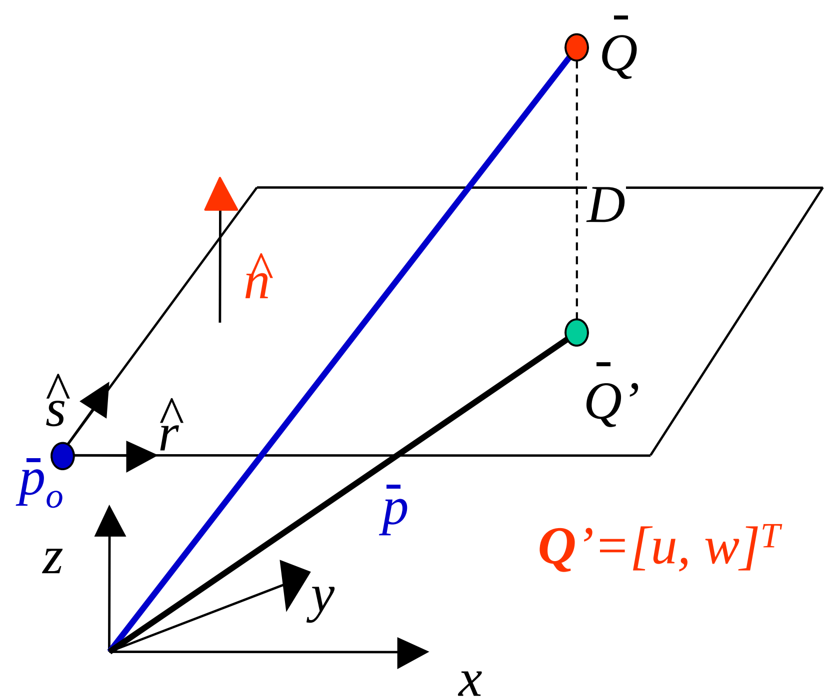


Solution

$$\because \; \overline{p} + \overline{QQ'} = \overline{Q}$$

$$\therefore \; \overline{p_0} + u\,\hat{r} + w\,\hat{s} + D\,\hat{n} = \overline{Q}$$

$$\boxed{u\,\hat{r} + w\,\hat{s} + D\,\hat{n} = \overline{Q} - \overline{p_0}}$$



$$\begin{bmatrix} \hat{r} & \hat{s} & \hat{n} \end{bmatrix} \begin{bmatrix} u \\ w \\ D \end{bmatrix} = \overline{Q} - \overline{P_0} \qquad \begin{bmatrix} r_x & s_x & n_x \\ r_y & s_y & n_y \\ r_z & s_z & s_y \end{bmatrix} \begin{bmatrix} u \\ w \\ D \end{bmatrix} = \begin{bmatrix} x_q \\ y_q \\ z_q \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

Planar Surface (defined by plane equation and boundaries)

$$\begin{cases} x = x(u,w) \\ y = y(u,w) \\ z = z(u,w) = -\dfrac{D}{c} - \dfrac{B}{c} y(u,w) - \dfrac{A}{c} x(u,w) \end{cases}$$

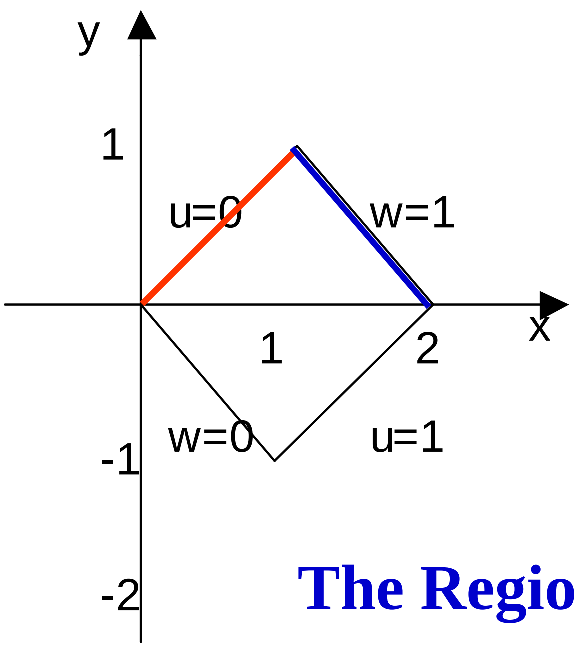
$Ax + By + Cz + D = 0$ is satisfied.

The two parametric equation *x(u,w)* & *y(u,w)* together with the boundaries specified by (*u=0, w=0,u=1,w=1*) determine the boundary of the projection of a surface $\bar{p}(u,w)$ in the *x*-*y* plane.

A Bounded Region of A Plane

$$\begin{cases} x = u + w \\ y = -u + w \\ z = \ldots\ldots\ldots \end{cases}$$

Chose the expressions of x and y**;**
z is determined by the plane equation.



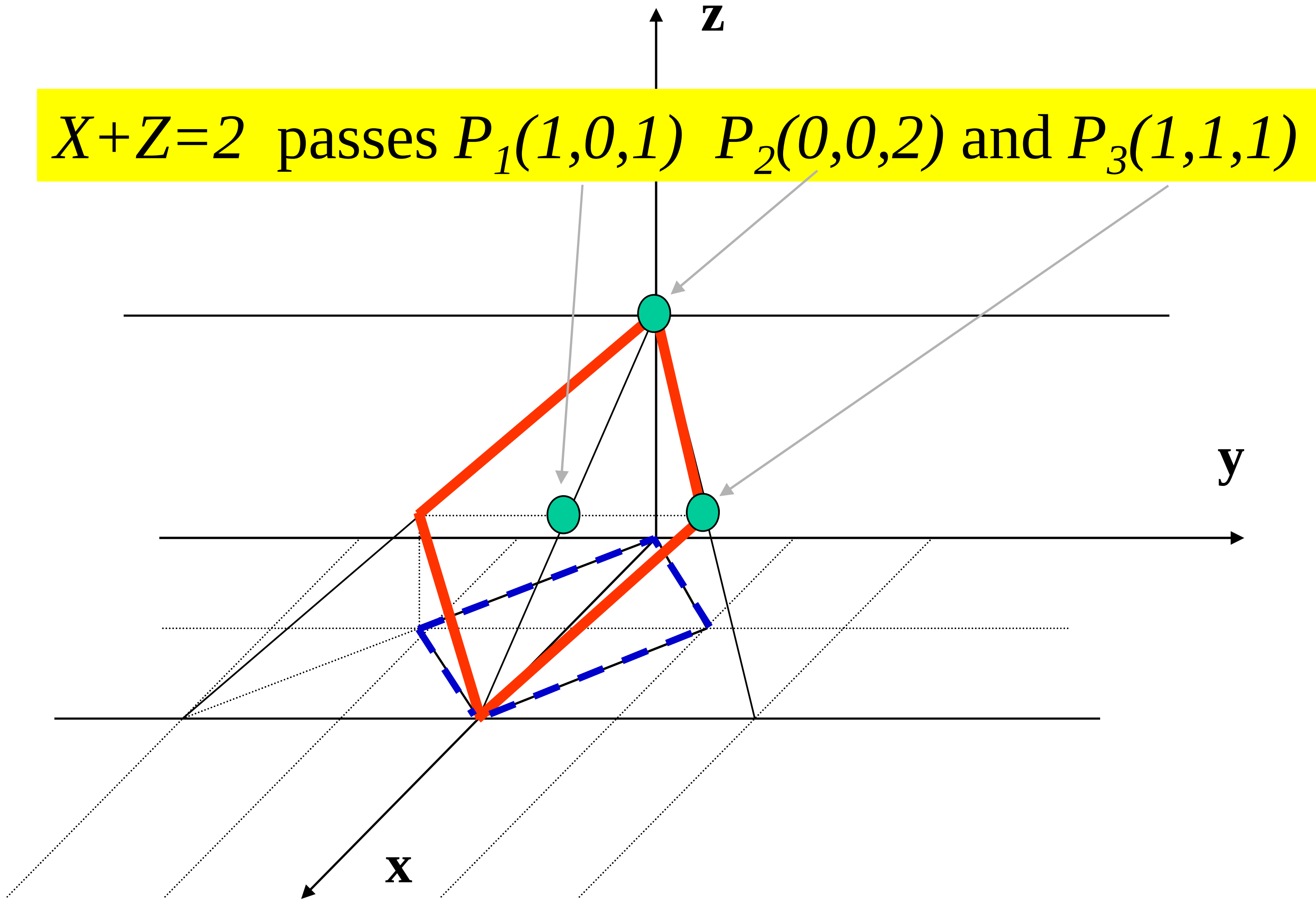
The Region

$u=0 \quad \begin{cases} x=w \\ y=w \end{cases} \quad y=x$

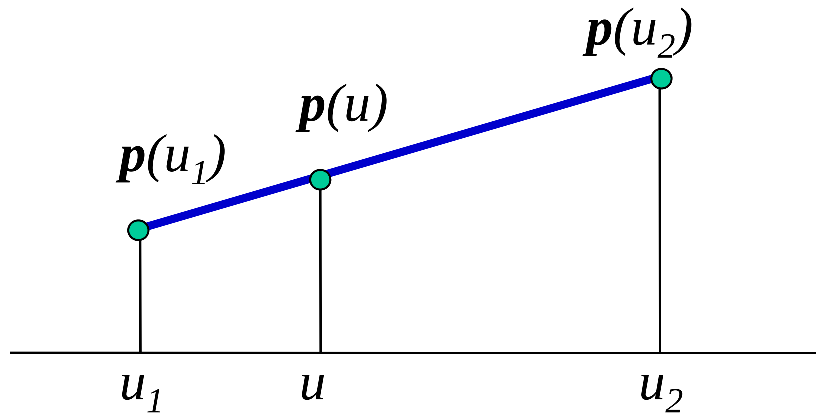
$u=1 \quad \begin{cases} x=1+w \\ y=-1+w \end{cases} \quad y=x-2$

$w=0 \quad \begin{cases} x=u \\ y=-u \end{cases} \quad y=-x$

$w=1 \quad \begin{cases} x=u+1 \\ y=-u+1 \end{cases} \quad y=-x+2$


$X+Z=2$ passes $P_1(1,0,1)$ $P_2(0,0,2)$ and $P_3(1,1,1)$
z
y
x

Bilinear Surface



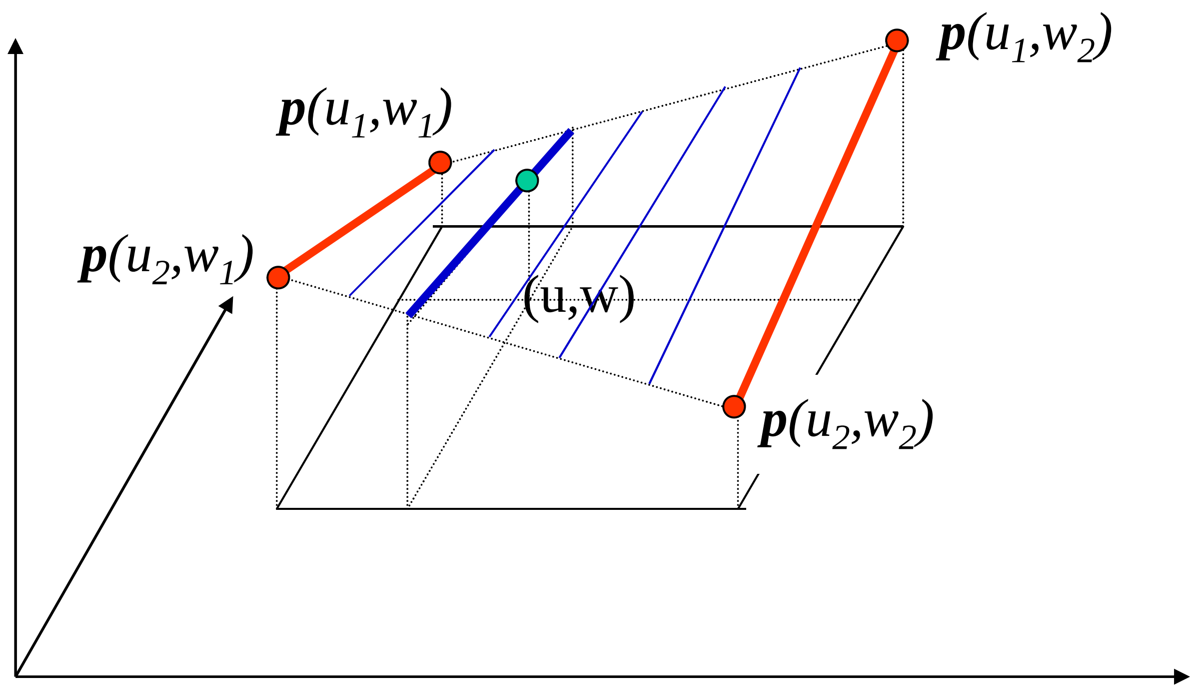
$$\because \quad \frac{u_1 - u}{u_1 - u_2} = \frac{p(u_1) - p(u)}{p(u_1) - p(u_2)}$$

$$p(u_1)(u_1 - u) - (u_1 - u)(p(u_1) - p(u_2)) = p(u)(u_1 - u_2$$

$$(u - u_2)\,p(u_1) + (u_1 - u)\,p(u_2) = p(u)(u_1 - u_2)$$

$$\therefore \quad p(u) = \frac{u_2 - u}{u_2 - u_1}\,p(u_1) + \frac{u - u_1}{u_2 - u_1}\,p(u_2)$$

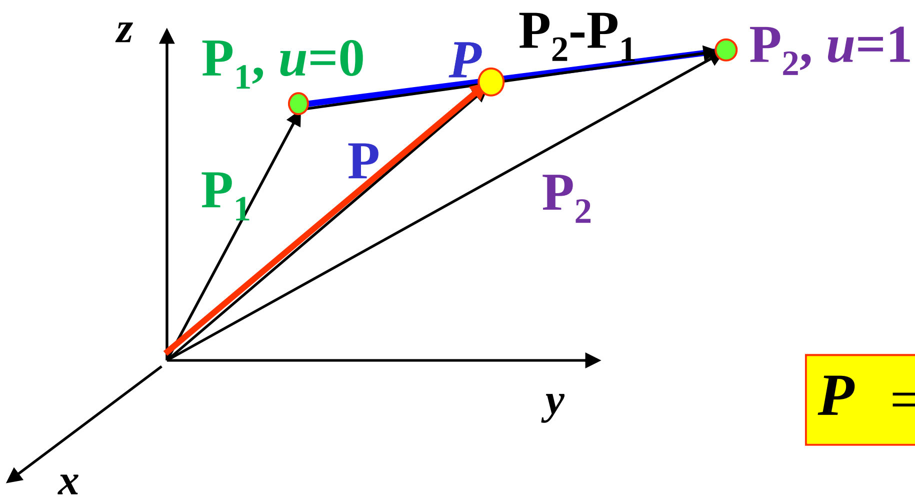
Bilinear Surface



$$\bar{p}(u,w) = \bar{p}(u_1,w_1)\left[\frac{u_2 - u}{u_2 - u_1}\right]\left[\frac{w_2 - w}{w_2 - w_1}\right] + \bar{p}(u_1,w_2)\left[\frac{u_2 - u}{u_2 - u_1}\right]\left[\frac{w - w_1}{w_2 - w_1}\right]$$

$$+ \bar{p}(u_2,w_1)\left[\frac{u - u_1}{u_2 - u_1}\right]\left[\frac{w_2 - w}{w_2 - w_1}\right] + \bar{p}(u_2,w_2)\left[\frac{u - u_1}{u_2 - u_1}\right]\left[\frac{w - w_1}{w_2 - w_1}\right]$$

Recall Parametric Representation of Lines



$$\boldsymbol{P} = \boldsymbol{P}_1 + (\boldsymbol{P} - \boldsymbol{P}_1)$$

$$\boldsymbol{P} - \boldsymbol{P}_1 = u\,(\boldsymbol{P}_2 - \boldsymbol{P}_1)$$

$$\boldsymbol{P} = \boldsymbol{P}_1 + u(\boldsymbol{P}_2 - \boldsymbol{P}_1), \quad 0 \le \mathrm{u} \le 1$$

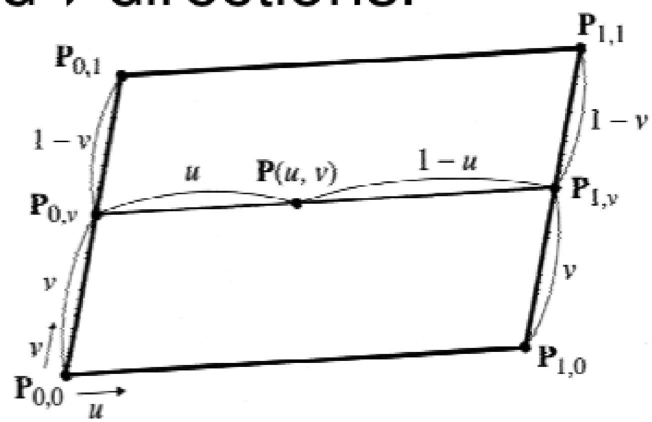
The Bilinear Surface

A bilinear surface is a linear interpolation of the four corner points in the u and v directions.

$$\mathbf{P}_{0,v} = (1-v)\mathbf{P}_{0,0} + v\mathbf{P}_{0,1}$$

$$\mathbf{P}_{1,v} = (1-v)\mathbf{P}_{1,0} + v\mathbf{P}_{1,1}$$

$$\mathbf{P}(u,v) = (1-u)\mathbf{P}_{0,v} + u\mathbf{P}_{1,v}$$



$$\mathbf{P}(u,v) = (1-u)[(1-v)\mathbf{P}_{0,0} + v\mathbf{P}_{0.1}] + u[(1-v)\mathbf{P}_{1,0} + v\mathbf{P}_{1,1}]$$

$$= [(1-u)(1-v) \quad u(1-v) \quad (1-u)v \quad uv]\begin{bmatrix}\mathbf{P}_{0,0}\\ \mathbf{P}_{1,0}\\ \mathbf{P}_{0,1}\\ \mathbf{P}_{1,1}\end{bmatrix} \quad (0 \le u \le 1, 0 \le v \le 1)$$

Ruled (or Lofted) Surfaces

Here we specify two of the four boundary curves, $\overline{p}(u,0)$ and $\overline{p}(u,1)$. These two curves can be defined by any of the methods that we discussed (cubic spline, Bezier, B-spline, NURBS, etc.). Points on the surface are obtained by linear interpolation.

$$\overline{p}(u,w) = \underline{\overline{p}(u,0)}(1-w) + \underline{\overline{p}(u,1)}w$$

or

$$\overline{p}(u,w) = \underline{\overline{p}(0,w)}(1-u) + \underline{\overline{p}(1,w)}u$$

If we choose straight lines for the boundaries, the ruled surface becomes a bilinear surface.

An Example

$$\overline{p}(u,w)=\overline{p}(u,0)(1-w)+\overline{p}(u,1)\,w$$

$\overline{p}(u,0)$ and $\overline{p}(u,1)$ are cubic splines with clamped ends

$$c_1=\overline{p}(u,0)$$

$p_1=[0\quad 0\quad 0]^T$

$p_2=[0\quad 1\quad 0]^T$

$p_1^{'}=[0\quad 1\quad 1]^T$

$p_2^{'}=[0\quad 1\quad 1]^T$

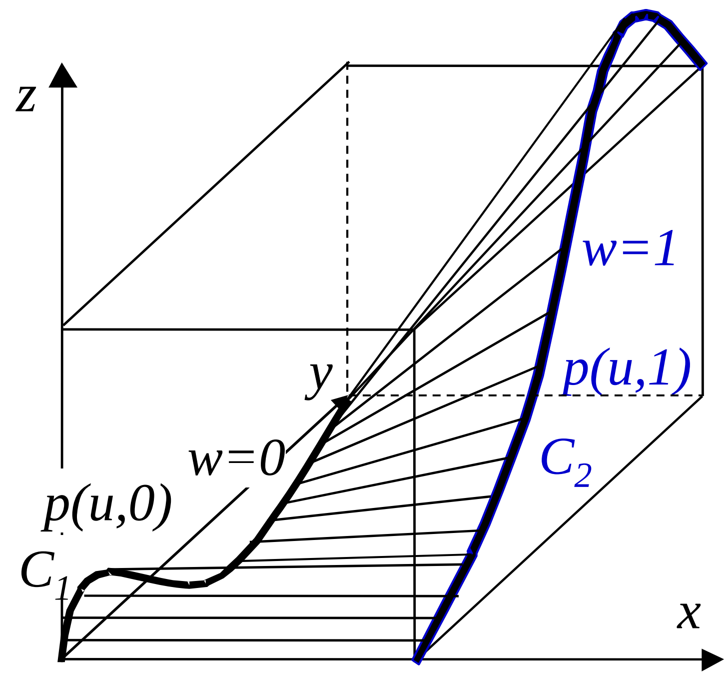
$$c_2=\overline{p}(u,1)$$

$p_1=[1\quad 0\quad 0]^T$

$p_2=[1\quad 1\quad 1]^T$

$p_1^{'}=[0\quad 1\quad 1]^T$

$p_2^{'}=[0\quad 1\quad -1]^T$



For each curve (C_1 and C_2 – *cubic splines*):

$$\vec{P}(u)=(2u^3-3u^2+1)\vec{P}_0+(-2u^3+3u^2)\vec{P}_1+(u^3-2u^2+u)\vec{P}_0^{'}+(u^3-u^2)\vec{P}_1^{'}$$

Substituting,

$$\bar{P}(u,0) = [0\ 0\ 0] + [0\ 1\ 1]u + [0\ 0\ -3]u^2 + [0\ 0\ 2]u^3$$

$$\bar{P}(u,1) = [1\ 0\ 0] + [0\ 1\ 1]u + [0\ 0\ 2]u^2 + [0\ 0\ -2]u^3$$

The equation for the surface is:

$$\bar{P}(u,w) = \bar{P}(u,0)(1-w) + \bar{P}(u,1)w$$

$$= \{[0\ 0\ 0] + [0\ 1\ 1]u + [0\ 0\ -3]u^2 + [0\ 0\ 2]u^3\}$$

$$\times(1-w) + \{[1\ 0\ 0] + [0\ 1\ 1]u$$

$$+ [0\ 0\ 2]u^2 + [0\ 0\ -2]u^3\}w$$

Rearranging in powers of u:

$$\bar{P}(u,w) = [w\ 0\ 0] + [0\ 1\ 1]u + [0\ 0\ (5w-3)]u^2 + [0\ 0\ (2-4w)]u^3$$

△ Display and Machining:

Given small Δu, & Δw.

$$u_{i+1} = u_i + \Delta u \qquad u_0 = 0$$

$$w_{i+1} = w_i + \Delta w \qquad w_0 = 0$$

calculate $\bar{P}(u_i, w_i) = \bar{P}_{ij}$ and generate a surface mesh.