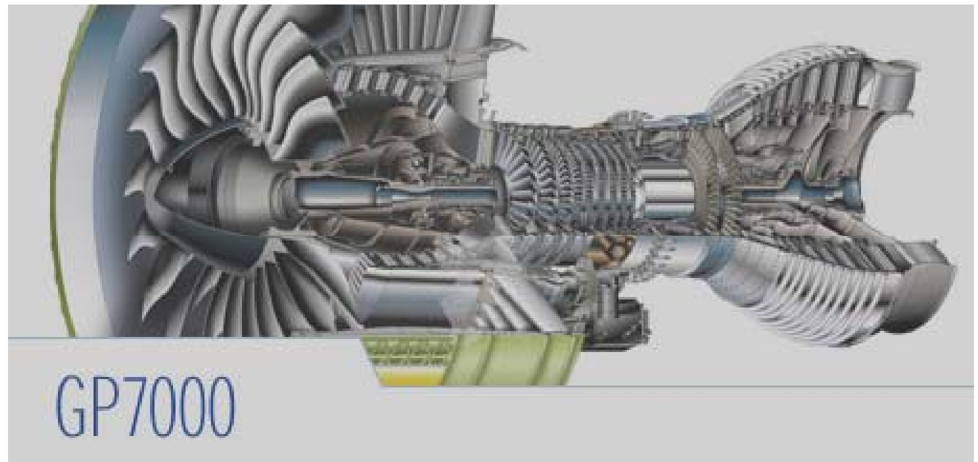
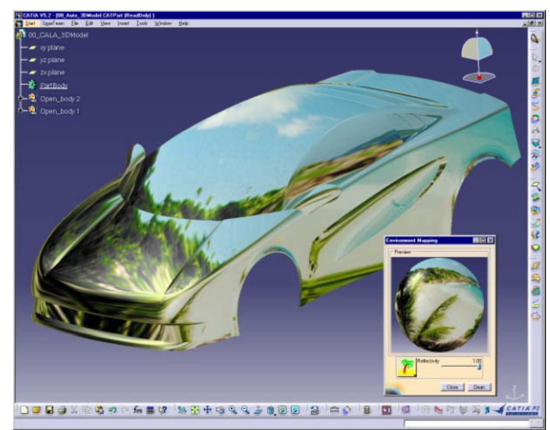
Free-form Surface II



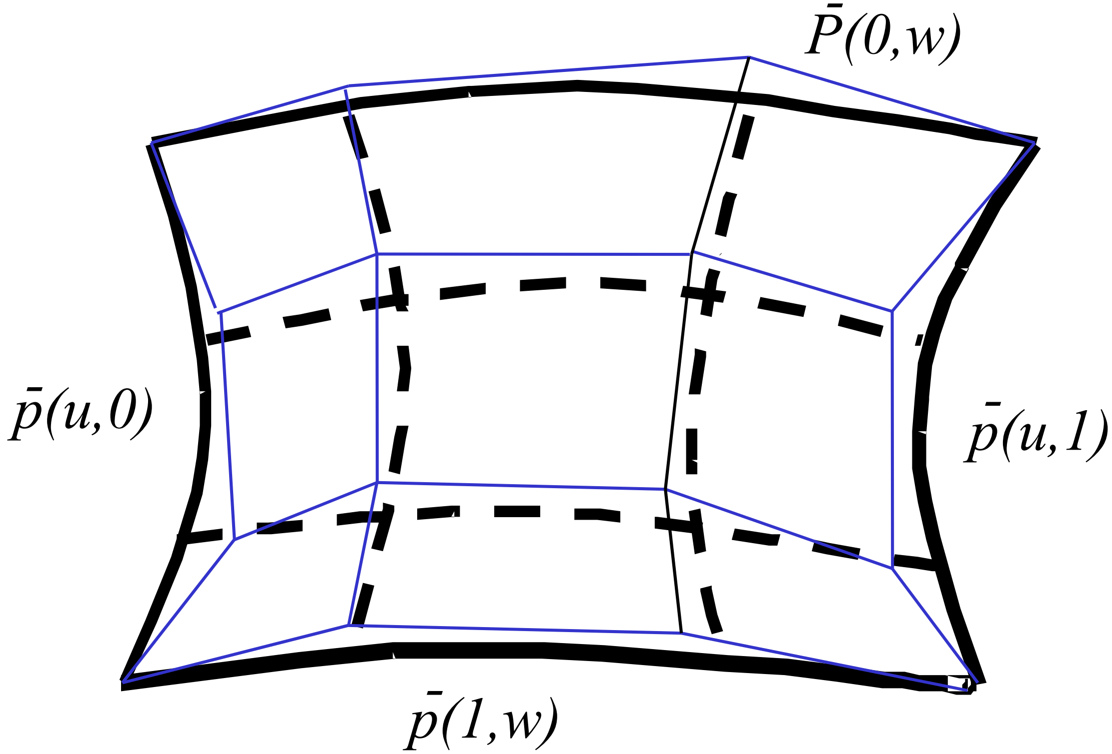


Type of Surfaces

- Planar Surface
- Bilinear Surface
- Ruled (lofted) Surface
- Bezier Surface
- Bi-cubic surface
- B-Spline Surface

Bezier Surface Patch

Bezier surfaces are formed by plotting families of Bezier curves. Changes of control points alter the global shape of the surface patch.



Bezier Surface Patch

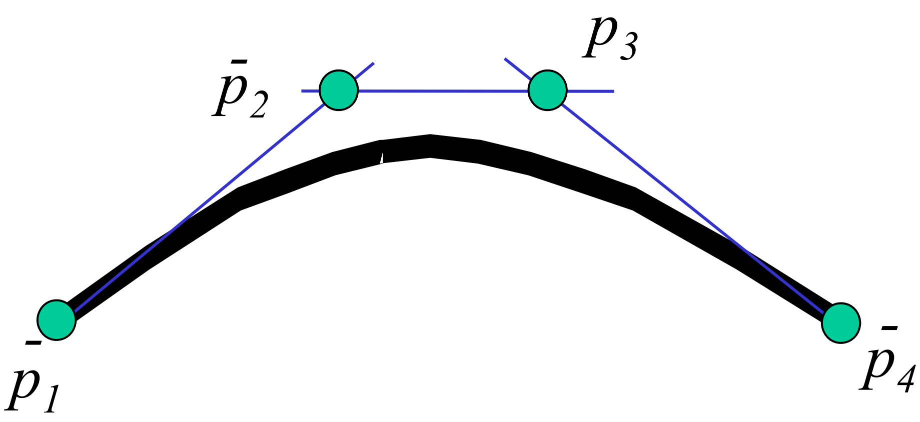
A Bezier Curve

$$\overline{p}(u) = \sum_{i=0}^{n} \overline{p_i} B_{i,n}(u) \qquad 0 \le u \le 1$$

$$B_{i,n}(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i}$$

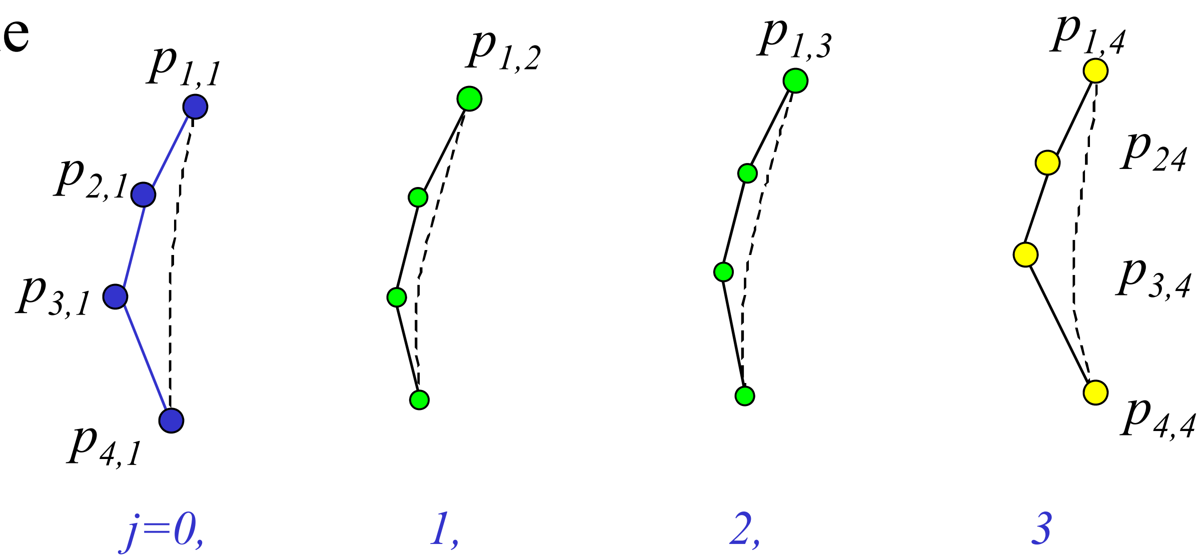
An Example

(4 points)



Bezier Surface Patch

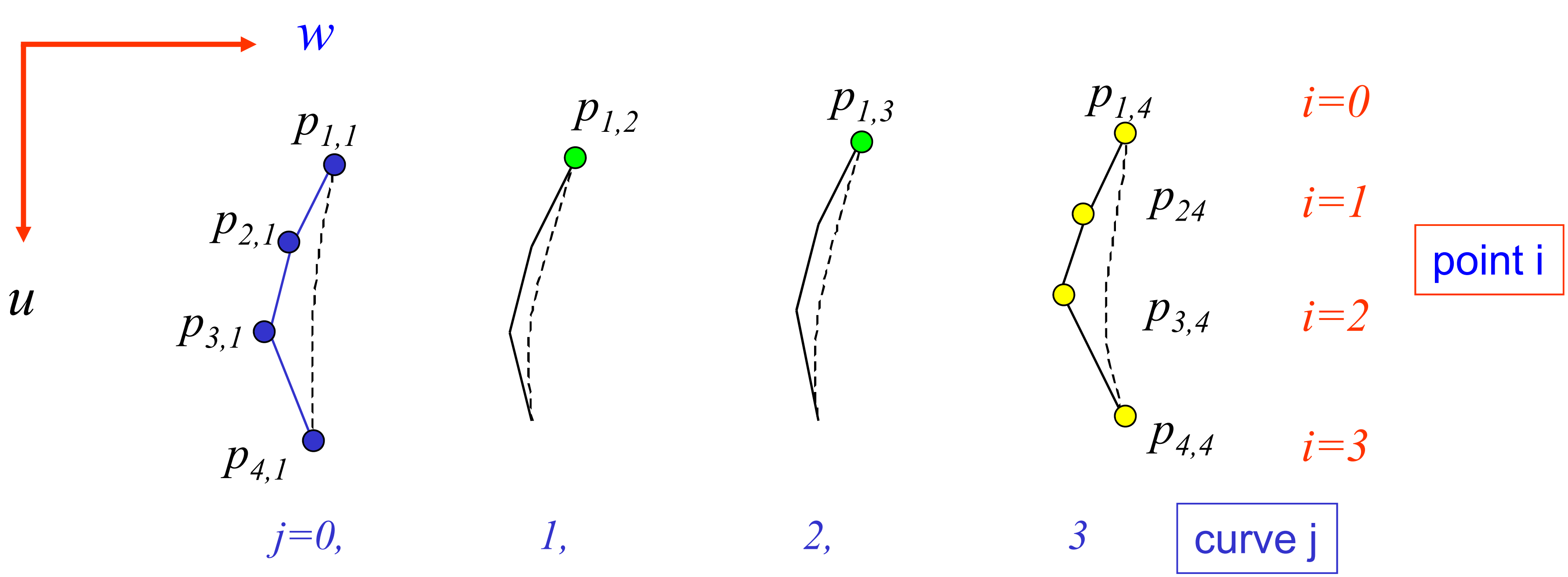
An Example



$$P = \begin{bmatrix} \overline{P_{1,1}} & \overline{P_{1,2}} & \overline{P_{1,3}} & \overline{P_{1,4}} \\ \overline{P_{2,1}} & \overline{P_{2,2}} & \overline{P_{2,3}} & \overline{P_{2,4}} \\ \overline{P_{3,1}} & \overline{P_{3,2}} & \overline{P_{3,3}} & \overline{P_{3,4}} \\ \overline{P_{4,1}} & \overline{P_{4,2}} & \overline{P_{4,3}} & \overline{P_{4,4}} \end{bmatrix} = \left\{ \overline{p_{i+1,j+1}} \right\}$$

curve 0 curve 1 curve 2 curve 3

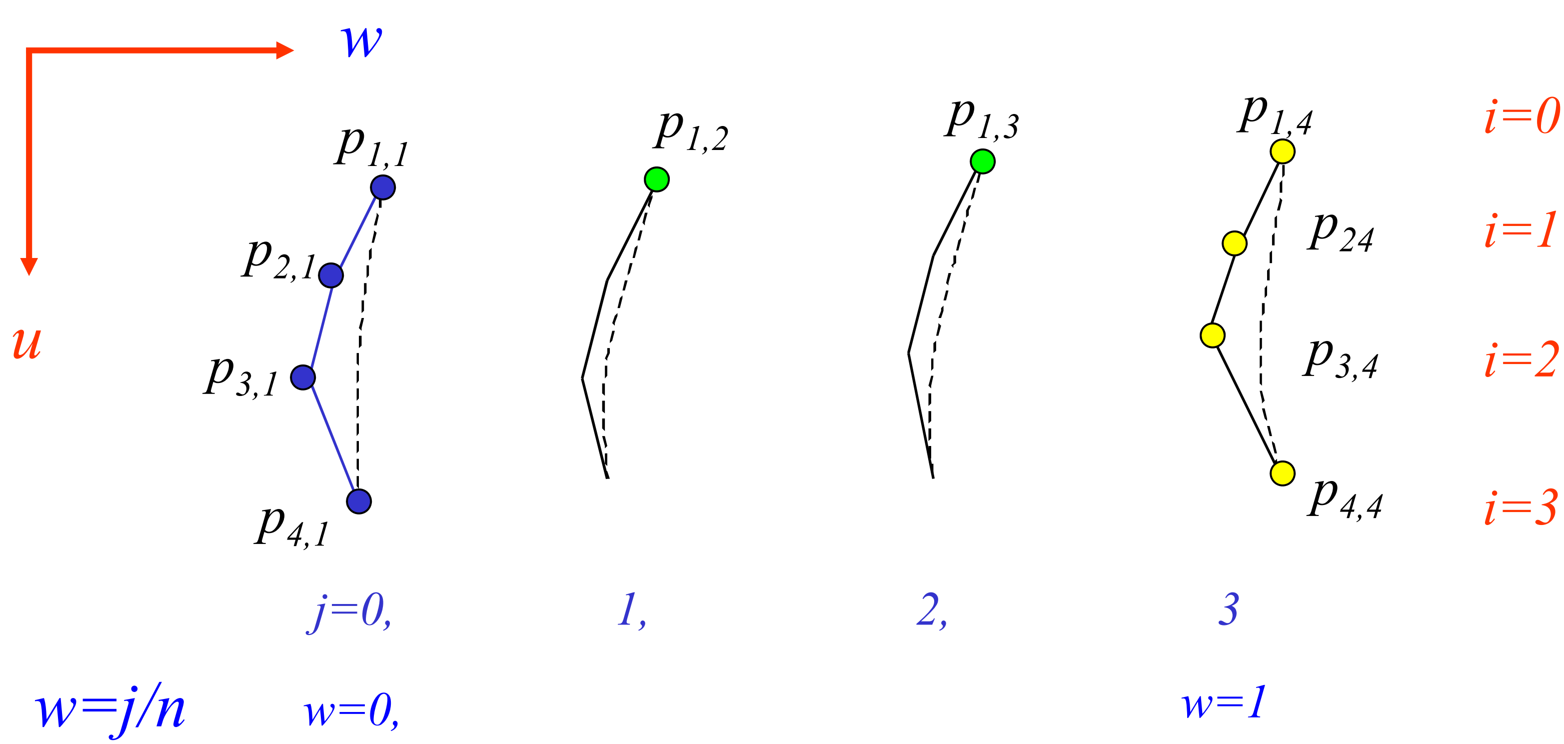
Bezier Surface Patch



$$\overline{p}(u,0)=\sum_{i=0}^{3}\overline{p_{i+1,1}}*B_{i,3}(u) \qquad (j=0 \quad or \quad w=0)$$

$$=(1-u)^3\,\overline{p_{1,1}}+3(1-u)^2u\,\overline{p_{2,1}}+3(1-u)u^2\,\overline{p_{3,1}}+u^3\,\overline{p_{4,1}}$$

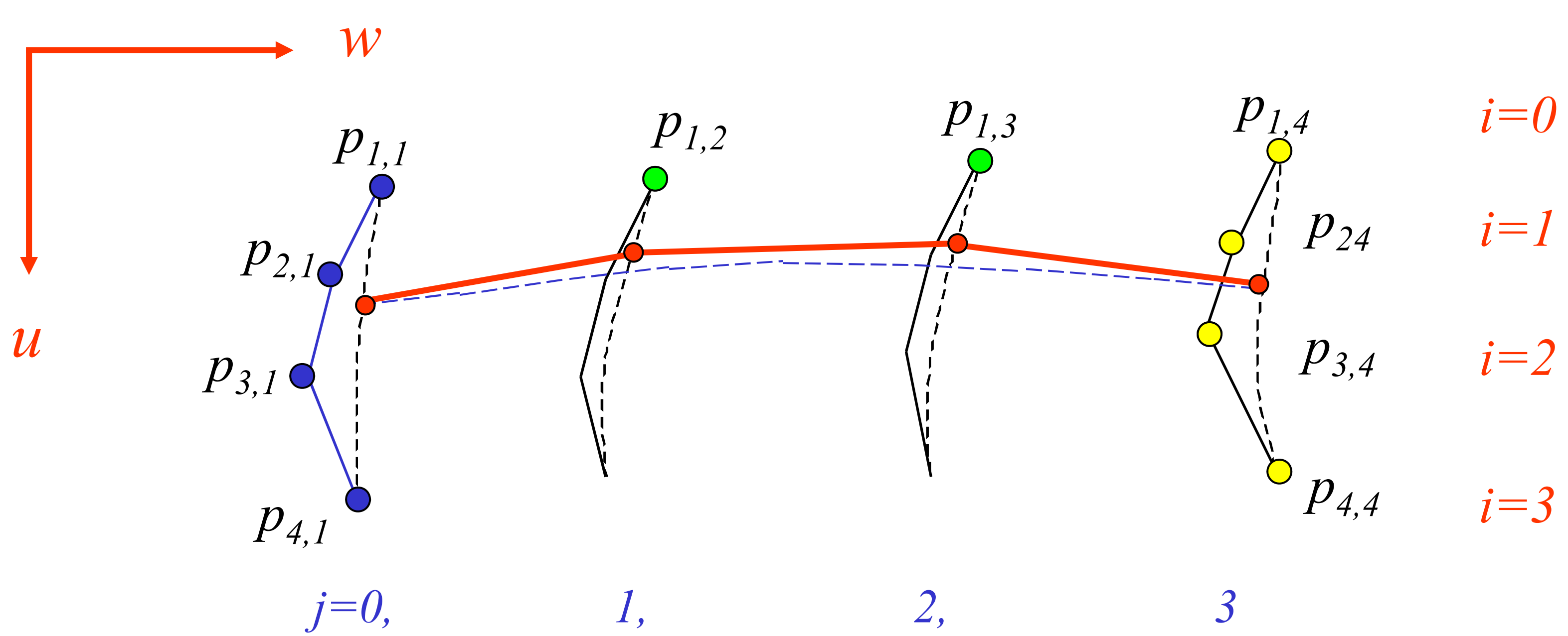
Bezier Surface Patch



For <u>other three</u> Bezier curves:

$$\overline{p}\left(u,\frac{j}{n}\right)=\sum_{i=0}^{3}\overline{p_{i+1,j+1}}*B_{i,3}(u)\qquad (j=1,\ 2,\ 3)$$

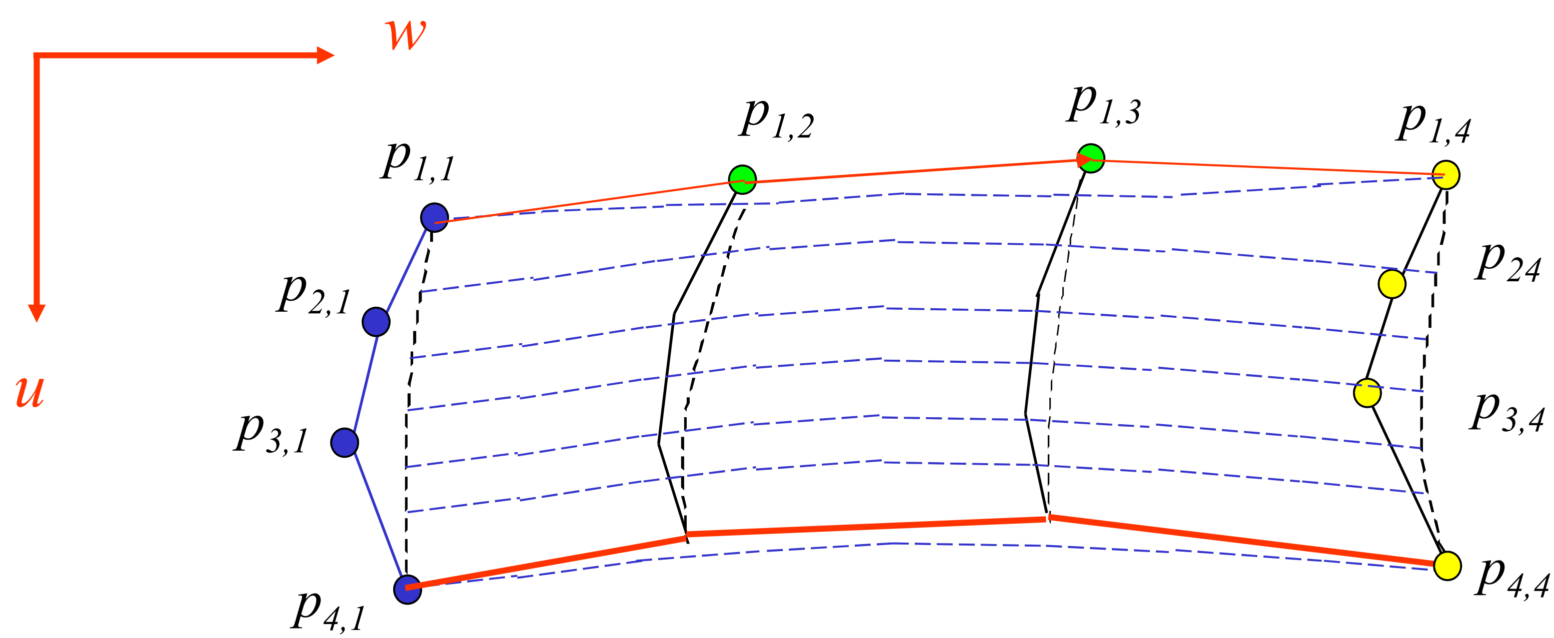
Bezier Surface Patch



Now we **chose a point on each curve** ($u=u_o$) to form a new control polygon, and generate a new Bezier curve.

Allow u to continuously change from $u=0$ to $u=1$, the Bezier surface is formed.

Bezier Surface Patch – Final Form



A Cubic Bezier Surface Patch

$$\overline{p}(u,w)=\sum_{j=0}^{3}\overline{p}\left(u,\frac{j}{n}\right)\times B_{j,3}(w)$$

$$\begin{cases}0\le u\le 1\\0\le w\le 1\end{cases}$$

n = 3
We have 4 points in each direction, cubic curves

$$=\sum_{j=0}^{3}\sum_{i=0}^{3}\overline{p_{i+1,j+1}}\times B_{i,3}(u)\times B_{j,3}(w)$$

$$=\begin{bmatrix}(1-u)^3 & 3(1-u)^2u & 3(1-u)u^2 & u^3\end{bmatrix}\begin{bmatrix}P_{11} & P_{12} & P_{13} & P_{14}\\P_{21} & P_{22} & P_{23} & P_{24}\\P_{31} & P_{32} & P_{33} & P_{34}\\P_{41} & P_{42} & P_{43} & P_{44}\end{bmatrix}\begin{bmatrix}(1-w)^3\\3(1-w)^2w\\3(1-w)w^2\\w^3\end{bmatrix}$$

$$=U^T[M_B][P][M_B]^T W$$

$$[M_B]=\begin{bmatrix}-1 & 3 & -3 & 1\\3 & -6 & 3 & 0\\-3 & 3 & 0 & 0\\1 & 0 & 0 & 0\end{bmatrix}$$

A General Bezier Surface Patch

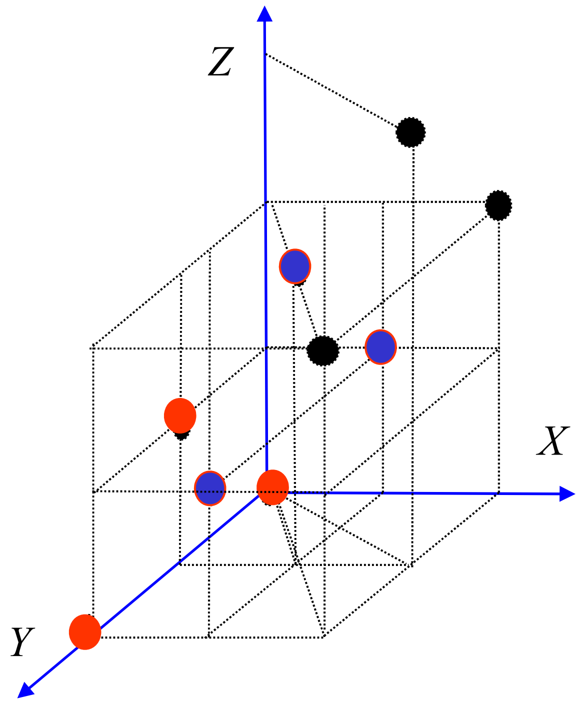
A general $n \times m$ Bezier surface:

$$\overline{p}(u,w) = \sum_{i=0}^{n} \sum_{j=0}^{m} \overline{p_{i+1,j+1}} B_{i,n}(u) B_{j,m}(w)$$

$$B_{i,n}(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i}$$

$$B_{j,m}(u) = \frac{m!}{j!(m-j)!} w^j (1-w)^{m-j}$$

An Example



A Bezier surface patch is specified by 9 control points:

$\boldsymbol{p}_{00} = (0,\ 0,\ 0)^T;\ \ \boldsymbol{p}_{01} = (0,\ 1,\ 1)^T;\ \ \boldsymbol{p}_{02} = (0,\ 2,\ 0)^T$

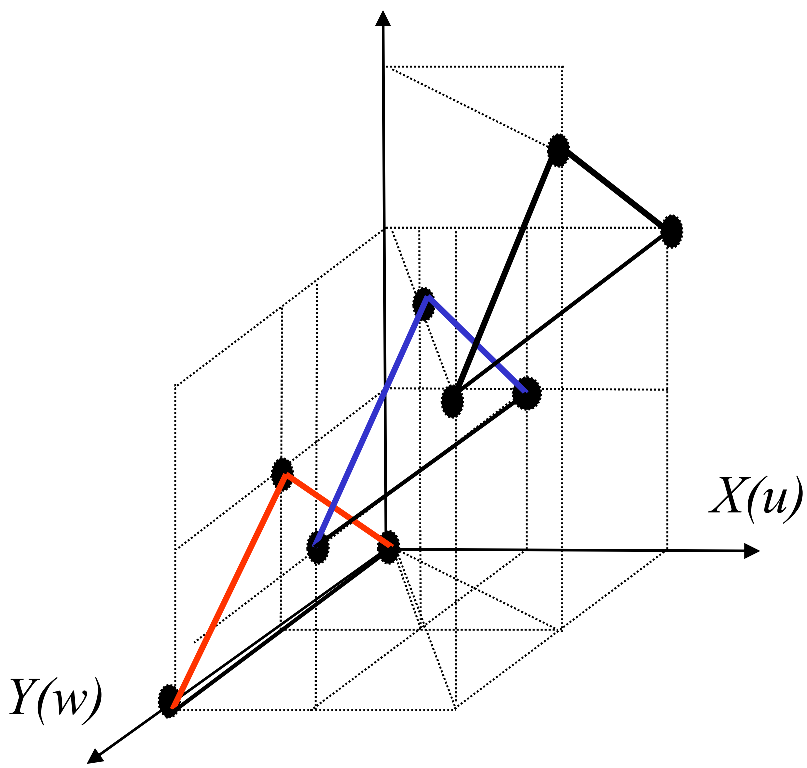
$\boldsymbol{p}_{10} = (1,\ 0,\ 1)^T;\ \ \boldsymbol{p}_{11} = (1,\ 1,\ 2)^T;\ \ \boldsymbol{p}_{12} = (1,\ 2,\ 1)^T$

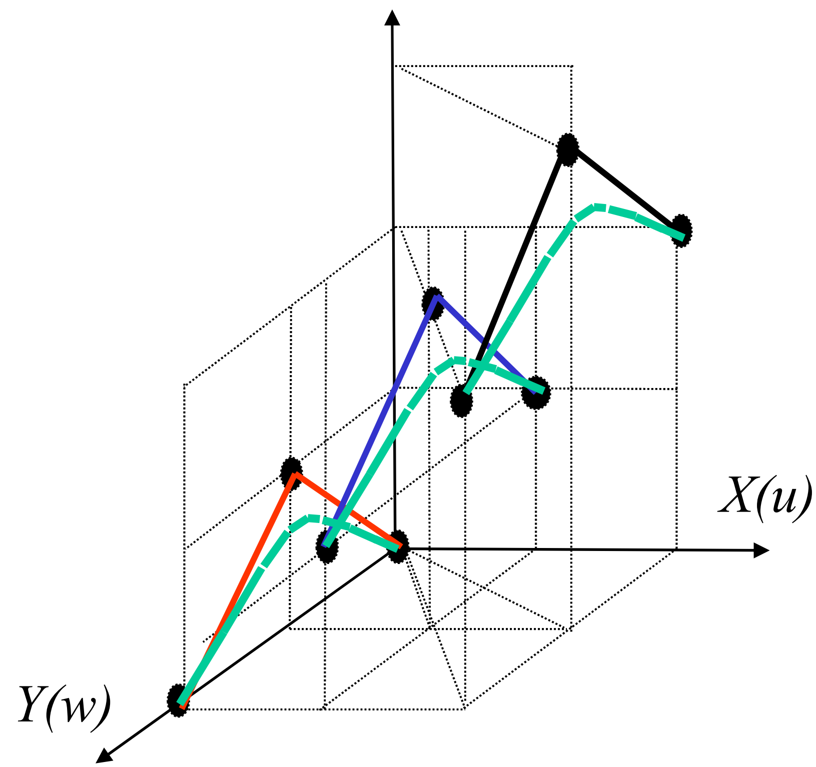
$\boldsymbol{p}_{20} = (2,\ 0,\ 2)^T;\ \ \boldsymbol{p}_{21} = (2,\ 1,\ 3)^T;\ \ \boldsymbol{p}_{22} = (2,\ 2,\ 2)^T$

1. Plot the control polygon and sketch the surface patch of $\boldsymbol{p}(u,w)$.
2. Given $\boldsymbol{p}_{00}\boldsymbol{p}_{01}\boldsymbol{p}_{02}$ as the $u = 0$ curve and $\boldsymbol{p}_{00}\boldsymbol{p}_{10}\boldsymbol{p}_{20}$ as the $w = 0$ curve, derive the Beizer curve expression for the two boundary curves, $\boldsymbol{q}(0,w)$ and $\boldsymbol{q}(u,0)$.
3. Derive the mathematical representation of the surface patch $\boldsymbol{p}(u,w)$.
4. Calculate $\boldsymbol{p}(0.5,\ 0.5)$

Solution

1. Sketch





2. If $\mathbf{p}_{00}\mathbf{p}_{01}\mathbf{p}_{02}$ defines the u=0 curve,

Solution

$$\vec{p}(u) = \sum_{i=0}^{n} \vec{p}_i B_{i,n}(u)$$

$$B_{i,n}(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i}$$

$$\vec{p}(u) = \underline{(1-u)^2} \vec{p}_0 + \underline{2u(1-u)} \vec{p}_1 + \underline{u^2} \vec{p}_2$$

$$P(0,w) = \underline{(1-w)^2} P_{00} + \underline{2w(1-w)} P_{01} + \underline{w^2} P_{02}$$
$$= [0 \quad 2w \quad 2w(1-w)]^T$$

If $\mathbf{p}_{00}\mathbf{p}_{10}\mathbf{p}_{20}$ defines the *w = 0* curve, similarly,

$$P(u,0) = \underline{(1-u)^2} P_{00} + \underline{2u(1-u)} P_{10} + \underline{u^2} P_{20}$$
$$= [2u \quad 0 \quad 2u)]^T$$

Solution

3. Surface Patch

$$P(u,w) = \sum_{j=0}^{2}\sum_{i=0}^{2} P_{i,j} * B_{i,2}(u) * B_{j,2}(w) \qquad \begin{cases} 0 \le u \le 1 \\ 0 \le w \le 1 \end{cases}$$

$$= \begin{bmatrix} (1-u)^2 & 2u(1-u) & u^2 \end{bmatrix} \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} (1-w)^2 \\ 2w(1-w) \\ w^2 \end{bmatrix}$$

$$\begin{aligned} P(u,w) = {} & (1-u)^2(1-w)^2 P_{00} + 2u(1-u)(1-w)^2 P_{10} + u^2(1-w)^2 P_{20} \\ & + 2(1-u)^2 w(1-w) P_{01} + 2u(1-u)\cdot 2w(1-w) P_{11} + u^2 \cdot 2w(1-w) P_{21} \\ & + (1-u)^2 w^2 P_{02} + 2u(1-u) w^2 P_{12} + u^2 w^2 P_{22} \end{aligned}$$

Solution

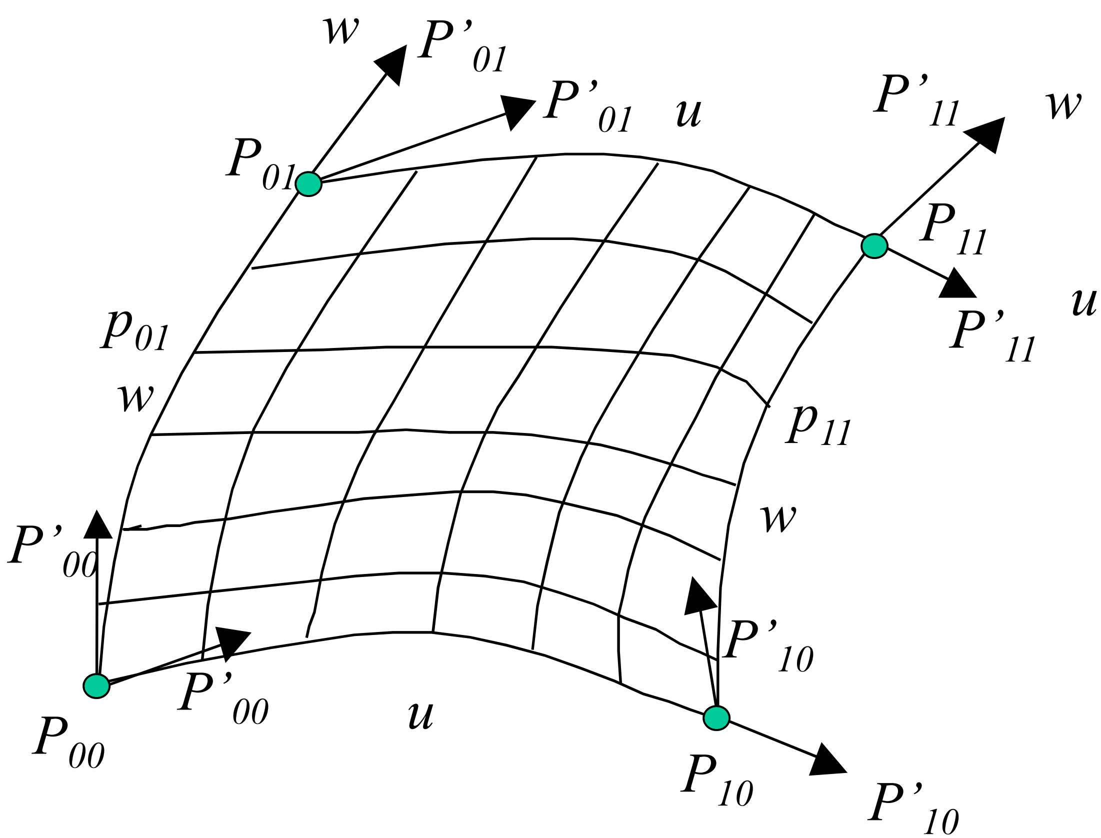
3. Calculate *P(0.5, 0.5)*

$$P(u,w) = (1-u)^2(1-w)^2 P_{00} + 2u(1-u)(1-w)^2 P_{10} + u^2(1-w)^2 P_{20}$$
$$+ 2(1-u)^2 w(1-w)P_{01} + 2u(1-u)\cdot 2w(1-w)P_{11} + u^2 \cdot 2w(1-w)P_{21}$$
$$+ (1-u)^2 w^2 P_{02} + 2u(1-u)w^2 P_{12} + u^2 w^2 P_{22}$$

$P(0.5, 0.5) = [\,1 \quad 1 \quad 1.5\,]^T$

X Y Z

Bi-Cubic Surface Patch



Bi-Cubic Surface Patch

A Cubic Spline

$$\vec{p}(u) = \sum_{i=0}^{3} \vec{C}_i u^i \qquad (0 \le u \le 1)$$

Unknowns:
 4x3 coefficients
Needed:
 2 end points (2x3)
 2 end slopes (2x3)

A Bi-cubic Surface Patch

$$\overline{p}(u,w) = \left[x(u,w), \quad y(u,w), \quad z(u,w) \right]^T$$

$$= \sum_{i=0}^{3} \sum_{j=0}^{3} \overline{a_{ij}} u^i w^j$$

Unknowns:
 16x3 coefficients
Needed:
 4 corner points (4x3)
 4x2 end slopes (8x3)
 4 twist vectors (4x3)

Bi-Cubic Surface Patch

$$\overline{p}(u,w) = [x(u,w), \quad y(u,w), \quad z(u,w)]^T = \sum_{i=0}^{3}\sum_{j=0}^{3}\overline{a_{ij}}u^i w^j$$

$$\overline{p}(u,w) = UAW^T = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} \overline{a_{33}} & \overline{a_{32}} & \overline{a_{31}} & \overline{a_{30}} \\ \overline{a_{23}} & \overline{a_{22}} & \overline{a_{21}} & \overline{a_{20}} \\ \overline{a_{13}} & \overline{a_{12}} & \overline{a_{11}} & \overline{a_{10}} \\ \overline{a_{03}} & \overline{a_{02}} & \overline{a_{01}} & \overline{a_{00}} \end{bmatrix} \begin{bmatrix} w^3 \\ w^2 \\ w \\ 1 \end{bmatrix}$$

$$A = M_H B M_H{}^T$$

$$B = \begin{bmatrix} [P] & [P_w] \\ [P_u] & [P_{uw}] \end{bmatrix} = \begin{bmatrix} \overline{p_{00}} & \overline{p_{01}} & \overline{p_{00}}^w & \overline{p_{01}}^w \\ \overline{p_{10}} & \overline{p_{11}} & \overline{p_{10}}^w & \overline{p_{11}}^w \\ \overline{p_{00}}^u & \overline{p_{01}}^u & \overline{p_{00}}^{uw} & \overline{p_{01}}^{uw} \\ \overline{p_{10}}^u & \overline{p_{11}}^u & \overline{p_{10}}^{uw} & \overline{p_{11}}^{uw} \end{bmatrix}$$

$$M_H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bi-Cubic Surface Patch

$$A = M_H B M_H^T$$

$$B = \begin{bmatrix} [P] & [P_w] \\ [P_u] & [P_{uw}] \end{bmatrix} = \begin{bmatrix} \overline{p_{00}} & \overline{p_{01}} & \overline{p_{00}}^w & \overline{p_{01}}^w \\ \overline{p_{10}} & \overline{p_{11}} & \overline{p_{10}}^w & \overline{p_{11}}^w \\ \overline{p_{00}}^u & \overline{p_{01}}^u & \overline{p_{00}}^{uw} & \overline{p_{01}}^{uw} \\ \overline{p_{10}}^u & \overline{p_{11}}^u & \overline{p_{10}}^{uw} & \overline{p_{11}}^{uw} \end{bmatrix}$$

$$M_H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

A Closer Look

4 corners

$$B=\begin{bmatrix}[P] & [P_w] \\ [P_u] & [P_{uw}]\end{bmatrix}=\begin{bmatrix}\overline{p}_{00} & \overline{p}_{01} & \overline{p}_{00}^{\,w} & \overline{p}_{01}^{\,w} \\ \overline{p}_{10} & \overline{p}_{11} & \overline{p}_{10}^{\,w} & \overline{p}_{11}^{\,w} \\ \overline{p}_{00}^{\,u} & \overline{p}_{01}^{\,u} & \overline{p}_{00}^{\,uw} & \overline{p}_{01}^{\,uw} \\ \overline{p}_{10}^{\,u} & \overline{p}_{11}^{\,u} & \overline{p}_{10}^{\,uw} & \overline{p}_{11}^{\,uw}\end{bmatrix}$$

u=0 curve

u=1 curve

twist vectors

w=0 curve

w=1 curve

Notes for Bi-cubic Surface Patch

- The surface patch is determined by 4 boundaries. Use $u, w = 0, 1$ to generate the boundary (interactively) of the surface.
- The four boundaries are cubic splines.
- Characteristics of the bi-cubic surface patch are very similar to those of the cubic spline, namely, lack of local control and the order of curve is fixed.
- Requirement of tangent and twist vectors as input data doesn't fit very well the design environment

B-Spline Surface

$$P(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{ij} \underline{N_{i,k}(u) N_{j,l}(v)}, \quad 0 \le u \le u_{\max}, \; 0 \le v \le v_{\max}$$

$$= [N_{0,k}(u) \;\; N_{1,k}(u) \;\; \cdots \;\; N_{n,k}(u)] \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0m} \\ P_{10} & P_{11} & \cdots & P_{1m} \\ \vdots & \vdots & & \vdots \\ P_{n0} & P_{n1} & \cdots & P_{nm} \end{bmatrix} \begin{bmatrix} N_{0,l}(v) \\ N_{1,l}(v) \\ \vdots \\ N_{m,l}(v) \end{bmatrix}$$

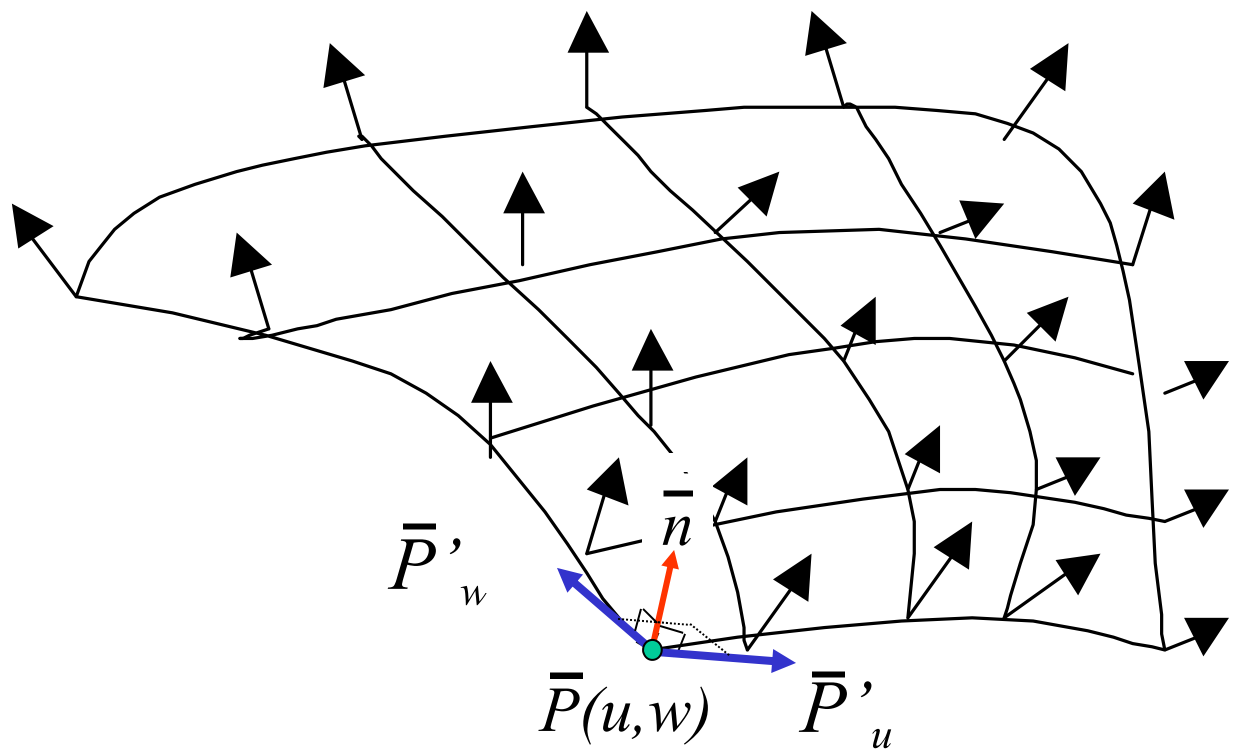
Features: local control and variation of degree

Surface Manipulations

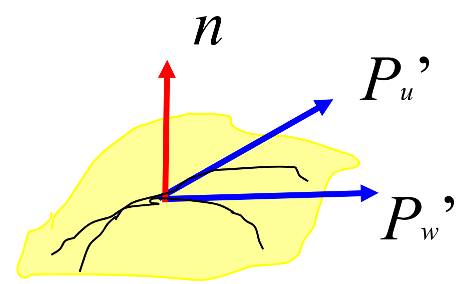
- Offset
- Blend
- Display
- Segmentation (division)
- Trimming
- Intersection
- Projection
- Transformation

Surface Normal

Applications: Direction, Distance Calculation, and Machining



Surface Offset

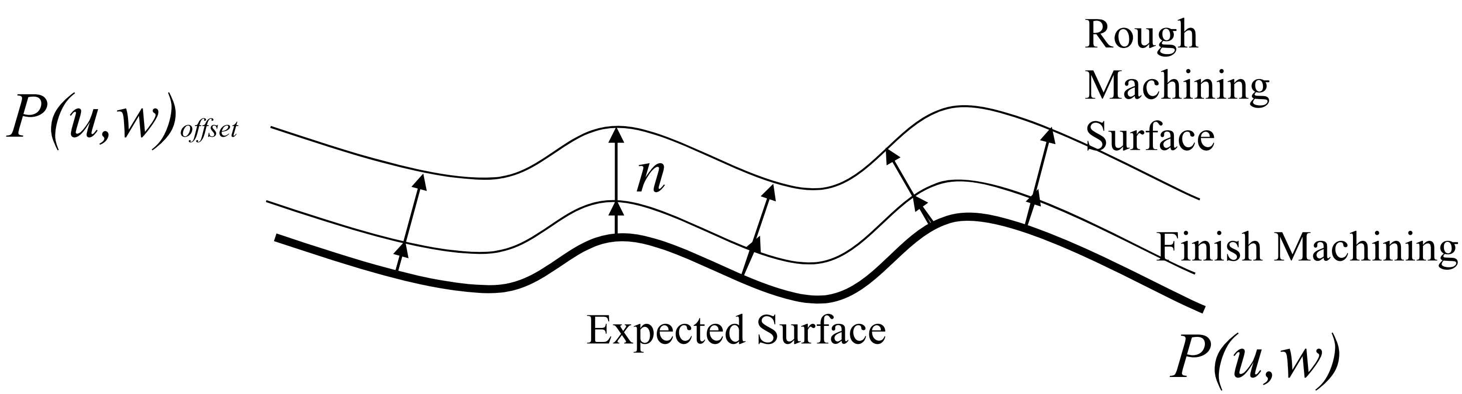


$$\overline{n}(u,w) = \overline{p_u^{'}}(u,w) \times \overline{p_w^{'}}(u,w)$$

$$\overline{p_u^{'}}(u,w) = \frac{\partial \overline{p}(u,w)}{\partial u}; \overline{p_w^{'}}(u,w) = \frac{\partial \overline{p}(u,w)}{\partial w}$$

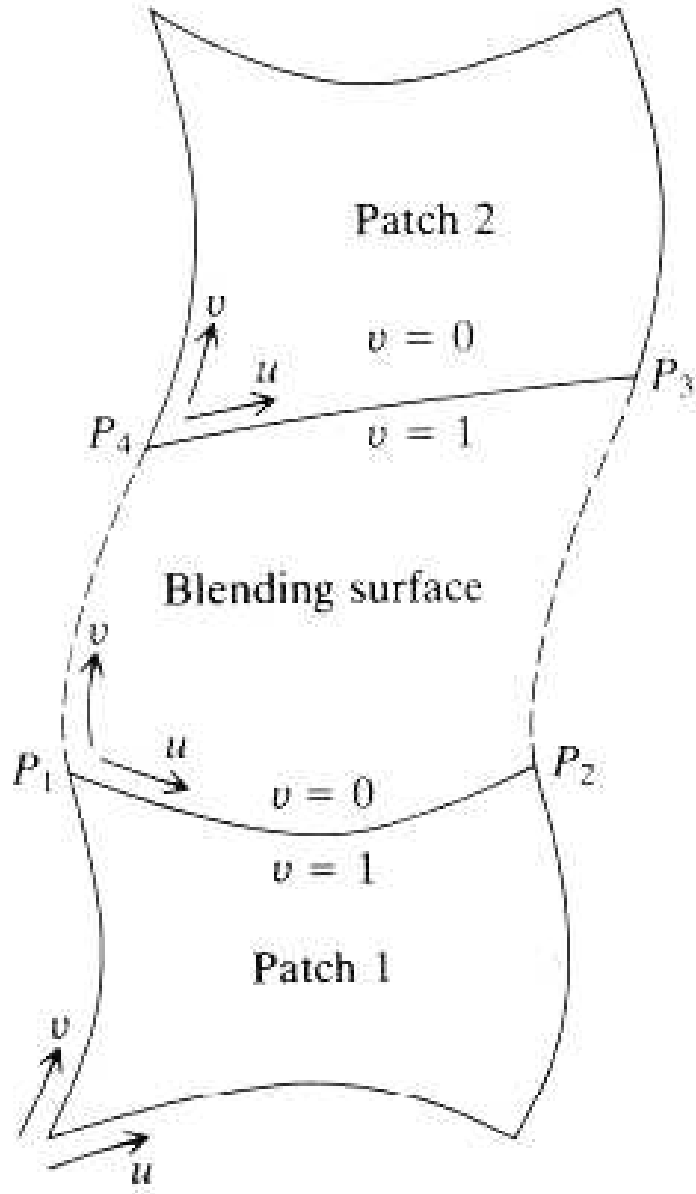
Generating the offset surface of a curved surface:

$$\overline{p}(u,w)_{offset} = \overline{p}(u,w) + \overline{n}(u,w) * d(u,w)$$

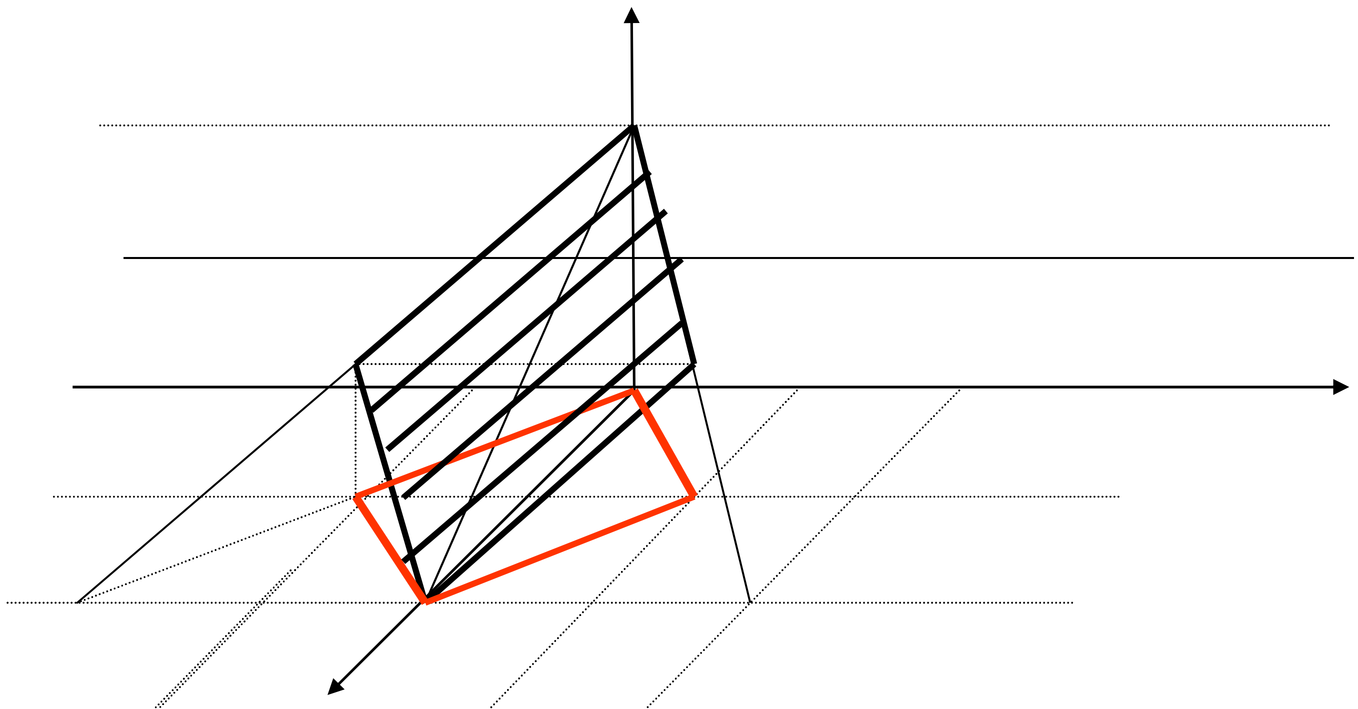


Blending Surfaces

A blending surface is a surface that connects two adjacent surfaces or patches. A blending surface is usually created for two given surface patches.



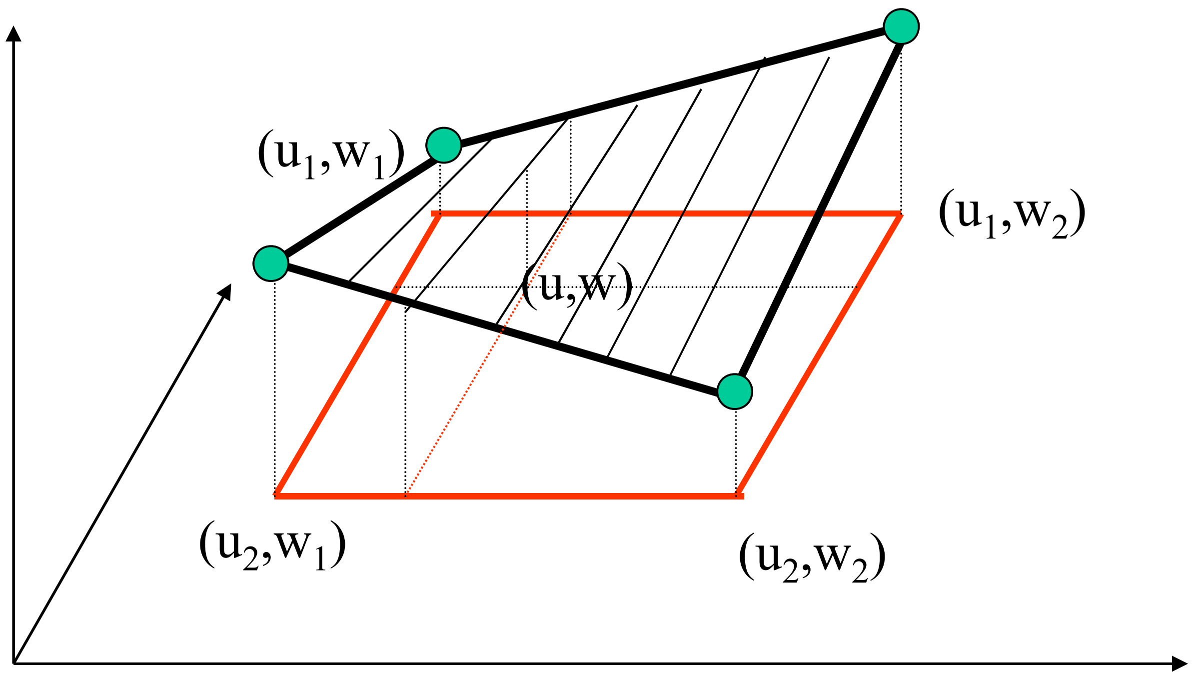
Review: A Plane (Patch)



$$\begin{cases} x = x(u,w) \\ y = y(u,w) \\ z = z(u,w) = -\frac{D}{c} - \frac{B}{c} y(u,w) - \frac{A}{c} x(u,w) \end{cases}$$

$$Ax + By + Cz + D = 0$$

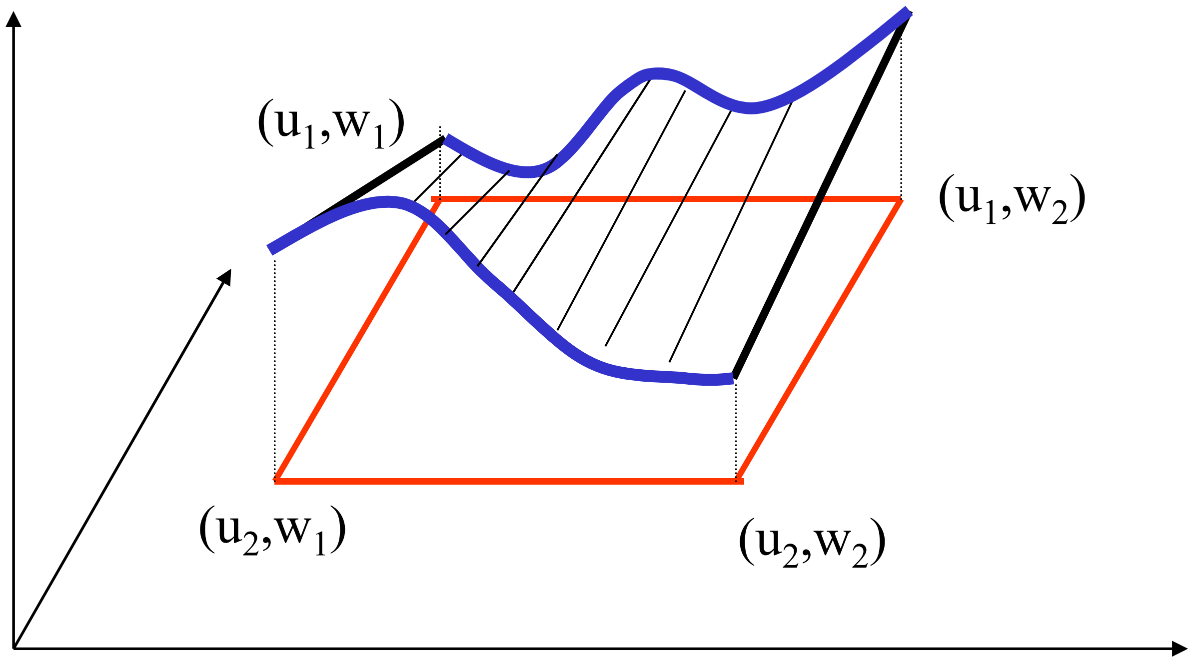
Review: Bilinear Surface



$$\vec{P}(u,w) = [1-u \quad u]\begin{bmatrix} \vec{p}(u_1,w_1) & \vec{p}(u_1,w_2) \\ \vec{p}(u_2,w_1) & \vec{p}(u_2,w_2) \end{bmatrix}\begin{bmatrix} 1-w \\ w \end{bmatrix}$$

$$u,w \in [0,1]$$

Review: Ruled (lofted) Surface

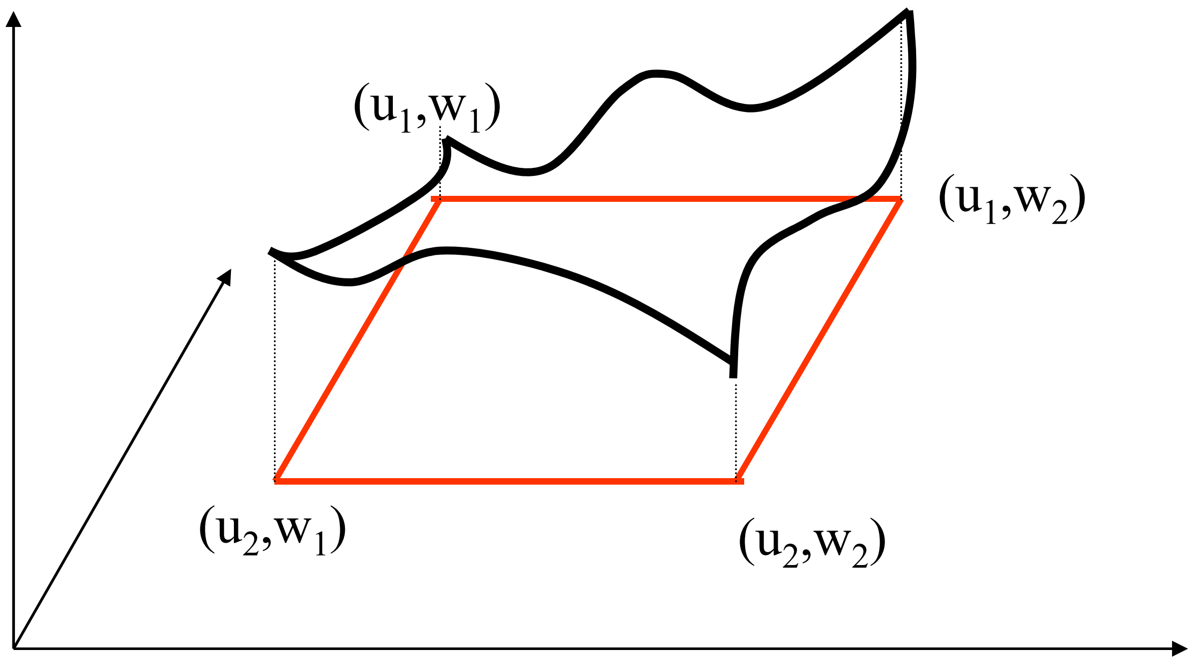


$$\vec{P}(u,w) = \vec{p}(u,0)(1-w) + \vec{p}(u,1)w$$

$$or\ \ \vec{P}(u,w) = \vec{p}(0,w)(1-u) + \vec{p}(1,w)u$$

$$u, w \in [0,1]$$

Review: Bezier Surface Patch



$$\vec{P}(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}\vec{p}_{i+1,j+1}B_{i,n}(u)B_{j,m}(w)\qquad u,w\in[0,1]$$

Summary

- **Planar, bilinear, and ruled surfaces**
- Cubic, **Bezier**, B-Spline Surfaces
  - Properties similar to corresponding curves
  - Curves are rudimental for surfaces
  - An extension to two dimension (u, w)
- Surface manipulation (offset, and blending)