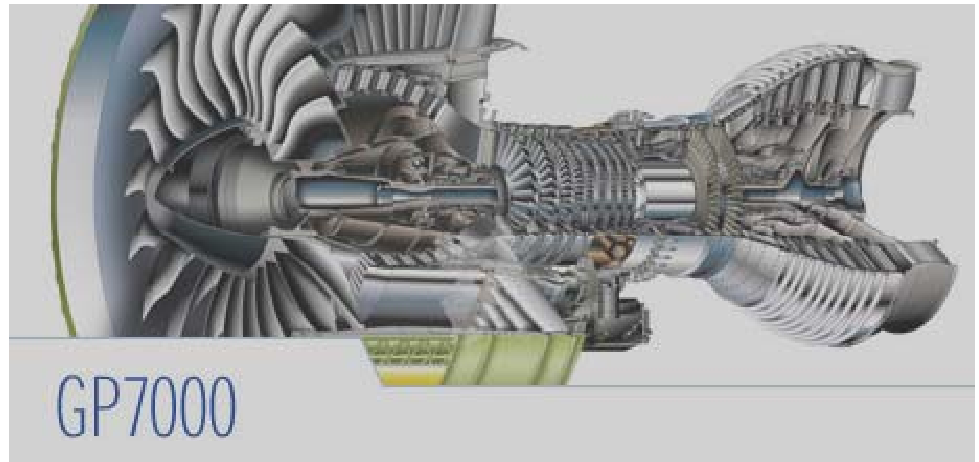
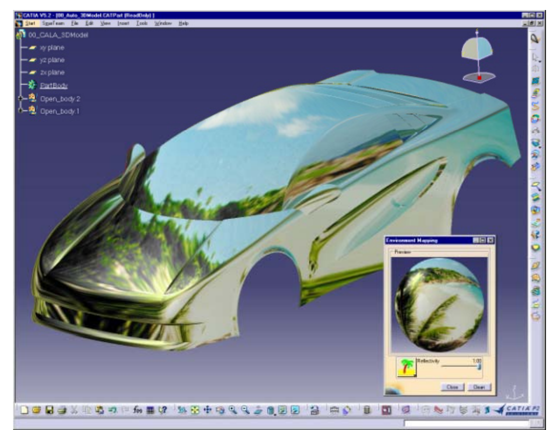
Free-form Surface I





Applications of Complex Surfaces

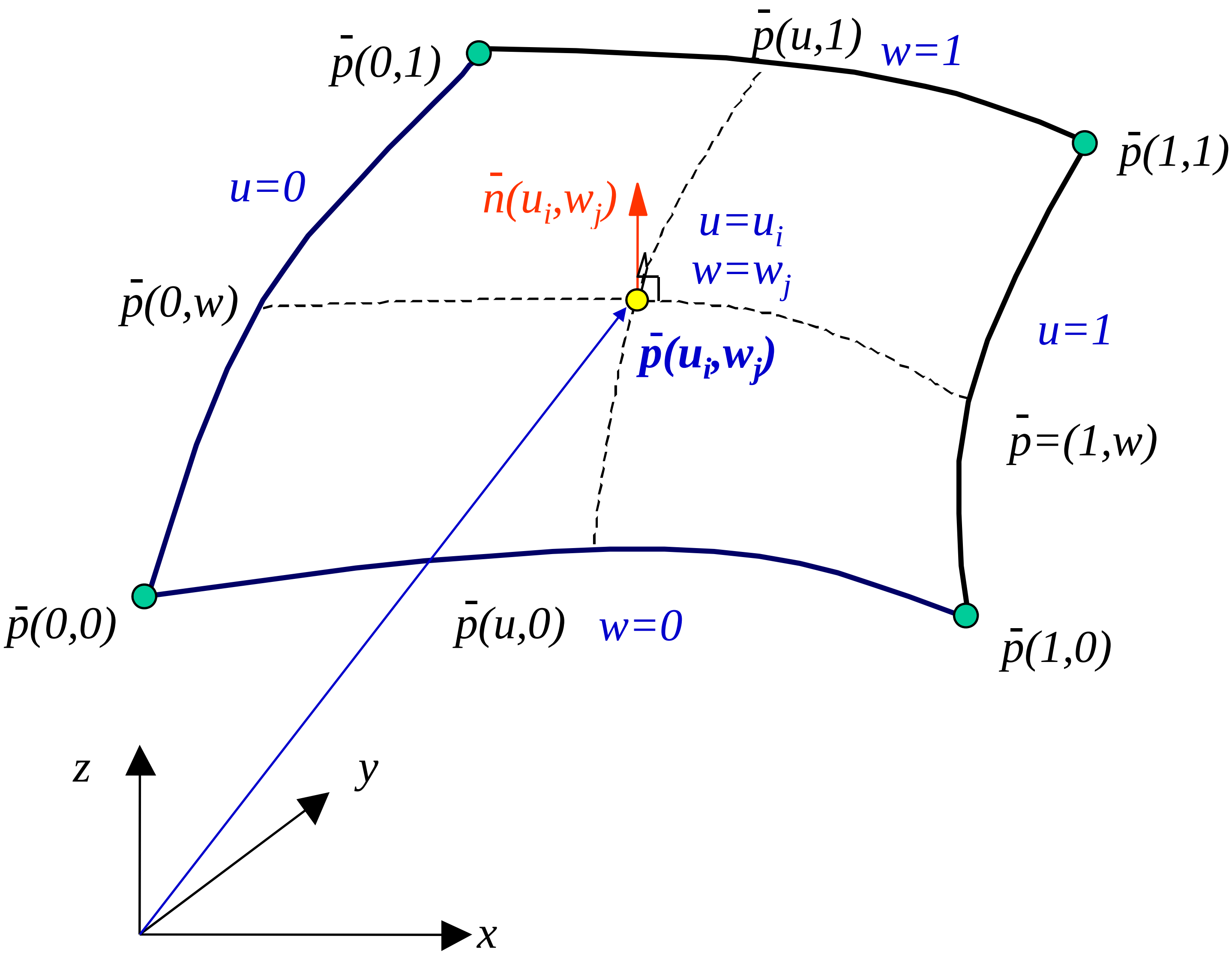
Defining and Controlling the Geometry of Curved Surface (Patch)

Surface Patch

A surface patch — a curved bounded collection of points whose coordinates are given by continuous, two-parameter, single-valued mathematical expression.

Function of the form:

$$\overline{p}(u,w) = \begin{bmatrix} x(u,w) & y(u,w) & z(u,w) \end{bmatrix}^T$$



p̄(0,1)
p̄(u,1)
w=1
p̄(1,1)
u=0
n̄(u_i,w_j)
u=u_i
w=w_j
p̄(0,w)
p̄(u_i,w_j)
u=1
p̄=(1,w)
p̄(0,0)
p̄(u,0)
w=0
p̄(1,0)
z
y
x

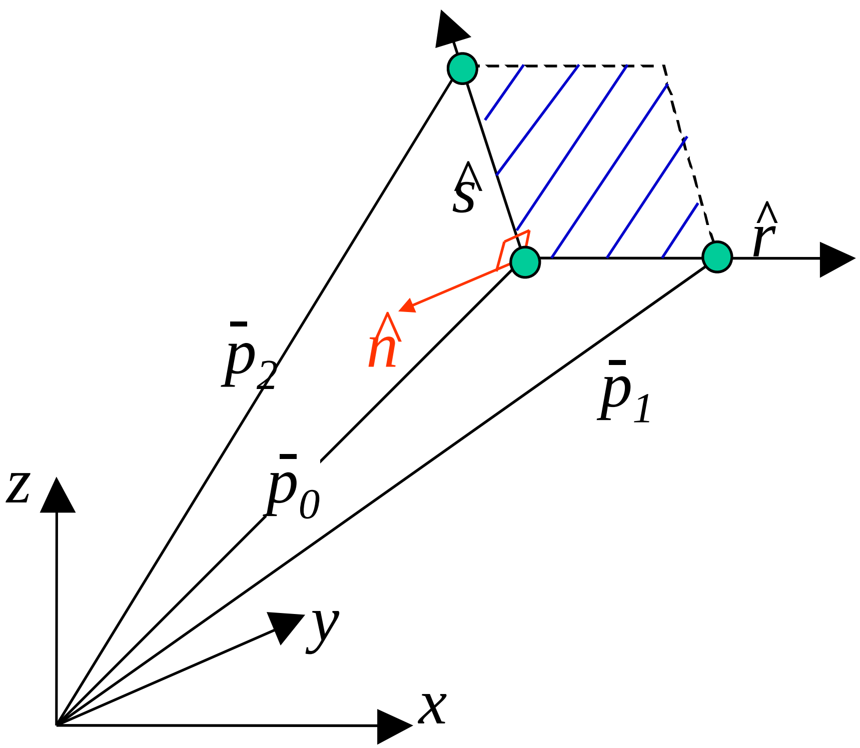
Type of Surface

- Planar Surface
- Bilinear Surface
- Ruled (Lofted) Surface
- Bi-cubic Surface
- Bezier Surface
- B-spline Surface

Planar Surface

- defined by three points and vectors

$$\overline{p}(u,w) = \overline{p_0} + u\left(\overline{p_1} - \overline{p_0}\right) + w\left(\overline{p_2} - \overline{p_0}\right) \qquad 0 \le u \le 1; \quad 0 \le w \le 1$$



Planar Surface

$$\overline{p}(u,w) = \overline{p_0} + u\left(\overline{p_1} - \overline{p_0}\right) + w\left(\overline{p_2} - \overline{p_0}\right) \qquad 0 \le u \le 1; \quad 0 \le w \le 1$$

$$\overline{p}(u,w) = \overline{p_0} + u\left|\overline{p_1} - \overline{p_0}\right|\hat{r} + w\left|\overline{p_2} - \overline{p_0}\right|\hat{s} \qquad 0 \le u \le 1; \quad 0 \le w \le 1$$

$$\hat{n} = \hat{r} \times \hat{s} \text{ — surface normal}$$

$$\hat{r} = \frac{\overline{p_1} - \overline{p_0}}{\left|\overline{p_1} - \overline{p_0}\right|}; \quad \hat{s} = \frac{\overline{p_2} - \overline{p_0}}{\left|\overline{p_2} - \overline{p_0}\right|}$$

Normalized Direction Vectors

Planar Surface

$$\overline{p}(u,w) = \overline{p_0} + u\left|\overline{p_1} - \overline{p_0}\right|\hat{r} + w\left|\overline{p_2} - \overline{p_0}\right|\hat{s} \qquad 0 \le u \le 1; \quad 0 \le w \le 1$$

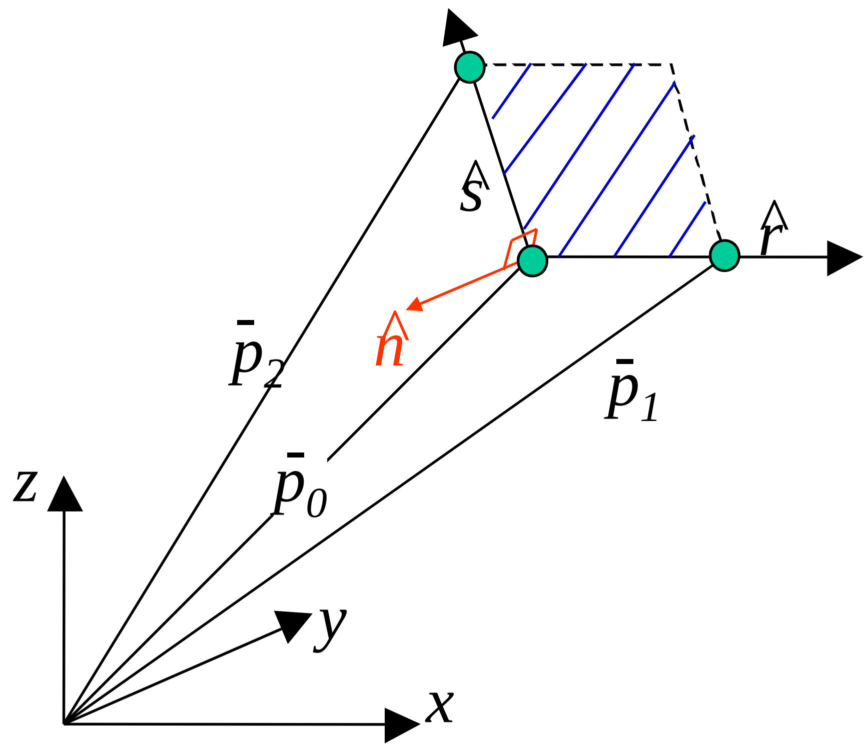
$$\boxed{\begin{aligned} Ax + By + Cz + D = 0 \\ \hat{n} = A\hat{i} + B\hat{j} + C\hat{k} \\ \hat{n} = \hat{r} \times \hat{s} \end{aligned}}$$

$$(\overline{P} - \overline{P_0}) \cdot \hat{n} = 0$$

$$(x - x_0) * A + (y - y_0) * B + (z - z_0) * C = 0$$

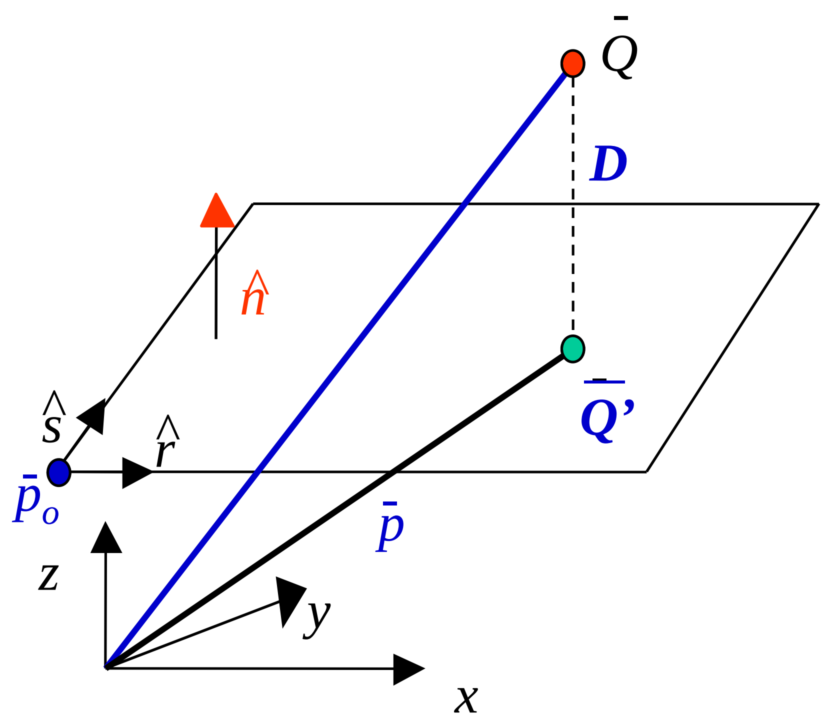
If define $D = -(Ax_0 + By_0 + Cz_0)$, then

$$Ax + By + Cz + D = 0$$



An Example

Find the distance between a point $\bar{Q} = \begin{bmatrix} x_q & y_q & z_q \end{bmatrix}^T$ and a plane $\bar{p} = \bar{p}_0 + u\hat{r} + w\hat{s} \quad (0 \le u \le 1,\ 0 \le w \le 1)$. That is to say, find the projection of point $\bar{Q}$ onto plane $\bar{P}$ and the distance *D*.

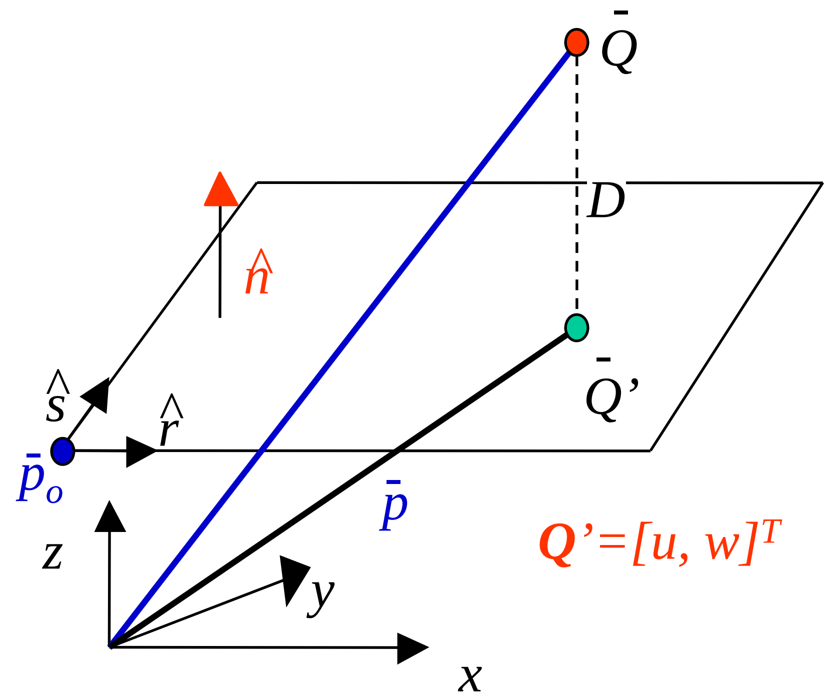


Solution

$$\because \overline{p} + \overline{QQ'} = \overline{Q}$$

$$\therefore \overline{p_0} + u\hat{r} + w\hat{s} + D\hat{n} = \overline{Q}$$

$$\boxed{u\hat{r} + w\hat{s} + D\hat{n} = \overline{Q} - \overline{p_0}}$$



$$\begin{bmatrix} \hat{r} & \hat{s} & \hat{n} \end{bmatrix} \begin{bmatrix} u \\ w \\ D \end{bmatrix} = \overline{Q} - \overline{P_0} \qquad \begin{bmatrix} r_x & s_x & n_x \\ r_y & s_y & n_y \\ r_z & s_z & s_y \end{bmatrix} \begin{bmatrix} u \\ w \\ D \end{bmatrix} = \begin{bmatrix} x_q \\ y_q \\ z_q \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

Planar Surface (defined by plane equation and boundaries)

$$\begin{cases} x = x(u,w) \\ y = y(u,w) \\ \boxed{z = z(u,w) = -\dfrac{D}{c} - \dfrac{B}{c} y(u,w) - \dfrac{A}{c} x(u,w)} \end{cases}$$

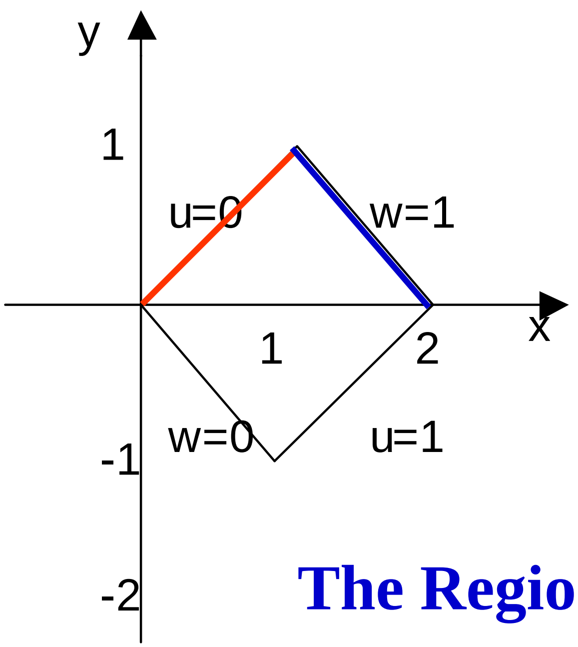
$\boxed{Ax + By + Cz + D = 0 \text{ is satisfied.}}$

The two parametric equation *x(u,w)* & *y(u,w)* together with the boundaries specified by (*u=0, w=0,u=1,w=1*) determine the boundary of the projection of a surface $\bar{p}(u,w)$ in the *x*-*y* plane.

A Bounded Region of A Plane

$$\begin{cases} x = u + w \\ y = -u + w \\ z = \ldots\ldots\ldots \end{cases}$$

Chose the expressions of *x* and *y*;
z is determined by the plane equation.



The Region

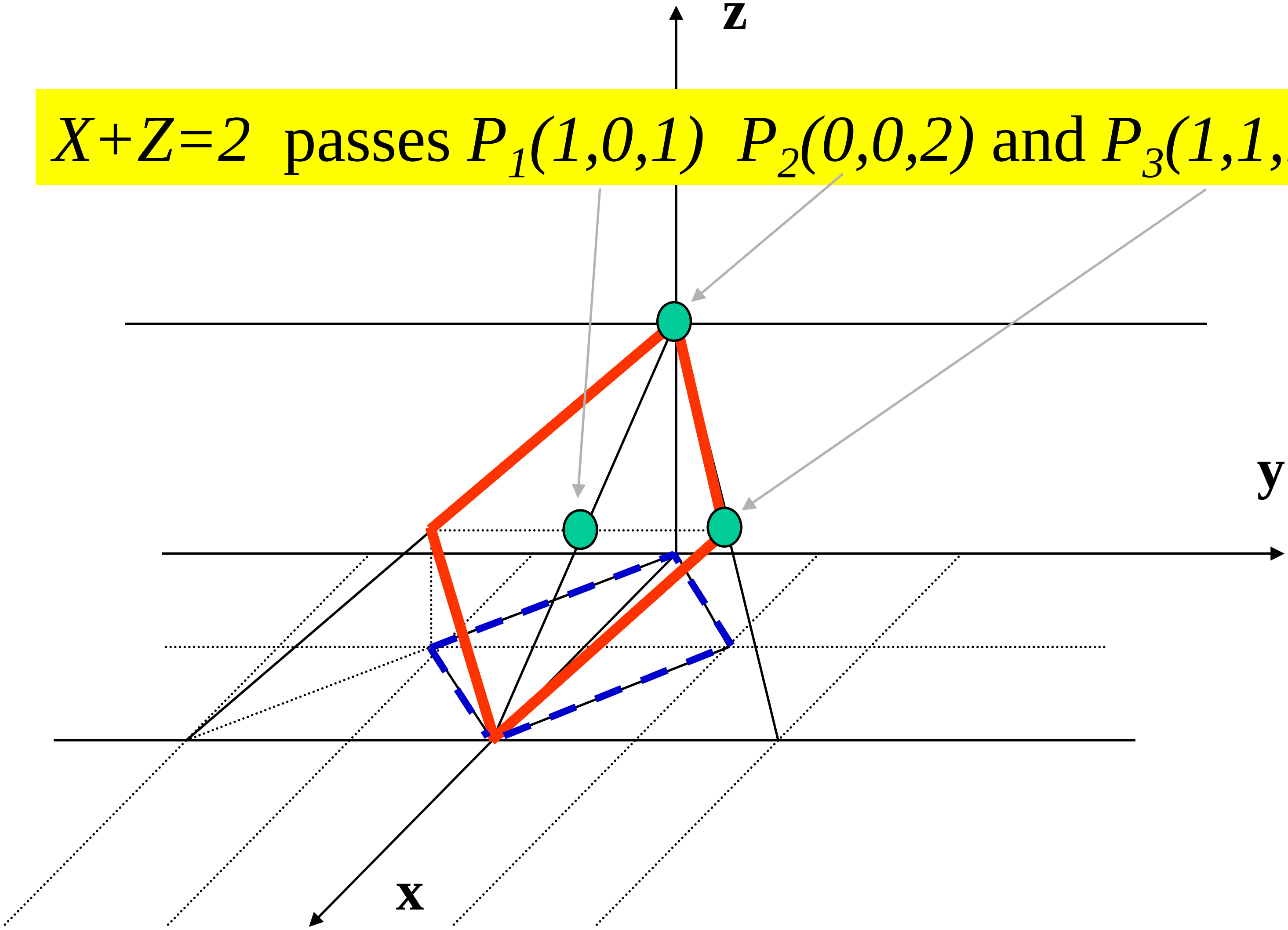
$$u=0 \quad \begin{cases} x=w \\ y=w \end{cases} \quad \boxed{y=x}$$

$$u=1 \quad \begin{cases} x=1+w \\ y=-1+w \end{cases} \quad y=x-2$$

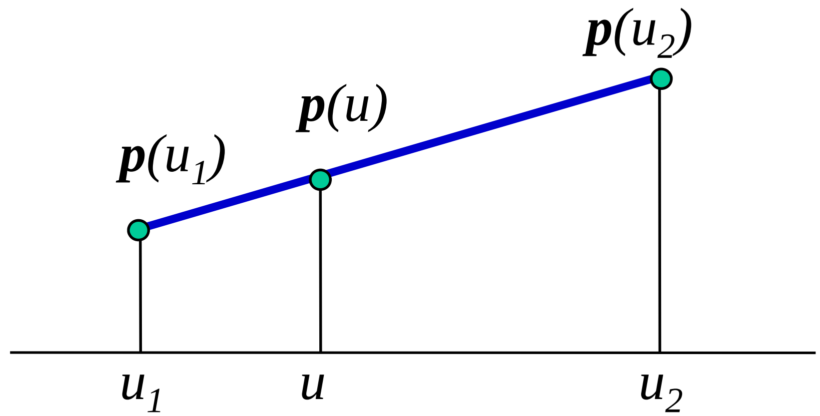
$$w=0 \quad \begin{cases} x=u \\ y=-u \end{cases} \quad y=-x$$

$$w=1 \quad \begin{cases} x=u+1 \\ y=-u+1 \end{cases} \quad \boxed{y=-x+2}$$

$X+Z=2$ passes $P_1(1,0,1)$ $P_2(0,0,2)$ and $P_3(1,1,1)$



Bilinear Surface



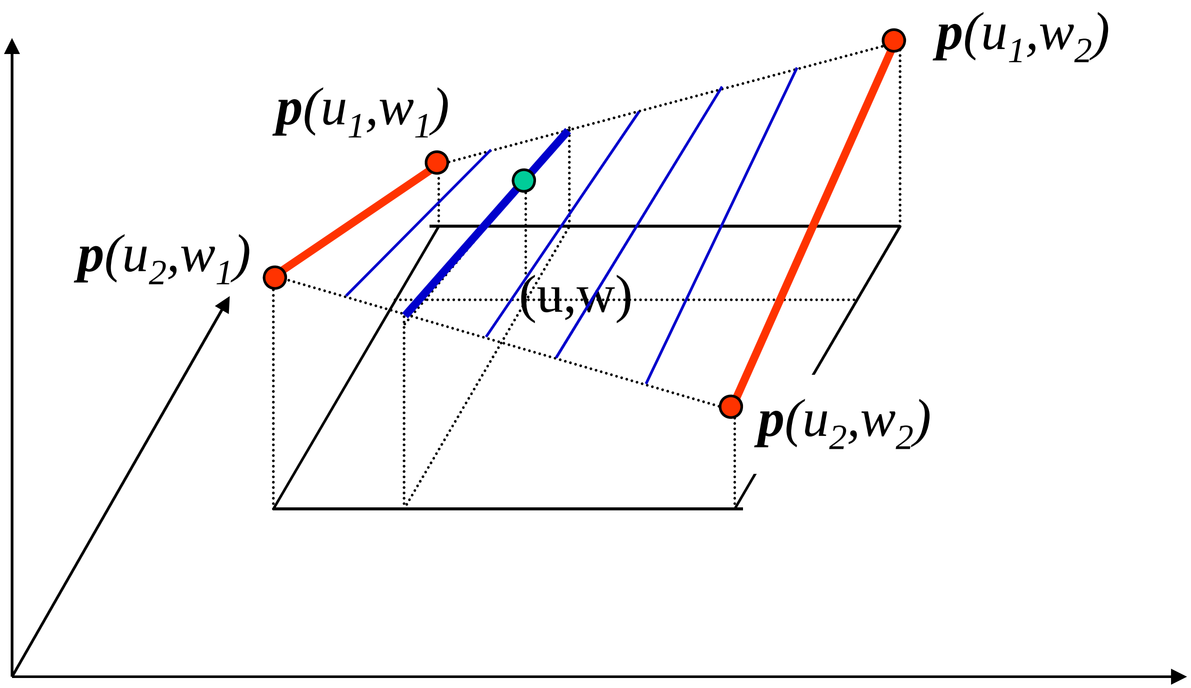
$$\because \quad \frac{u_1 - u}{u_1 - u_2} = \frac{p(u_1) - p(u)}{p(u_1) - p(u_2)}$$

$$p(u_1)(u_1 - u) - (u_1 - u)(p(u_1) - p(u_2)) = p(u)(u_1 - u_2$$

$$(u - u_2)\,p(u_1) + (u_1 - u)\,p(u_2) = p(u)(u_1 - u_2)$$

$$\boxed{\therefore \quad p(u) = \frac{u_2 - u}{u_2 - u_1}\,p(u_1) + \frac{u - u_1}{u_2 - u_1}\,p(u_2)}$$

Bilinear Surface



$$\bar{p}(u,w) = \bar{p}(u_1,w_1)\left[\frac{u_2 - u}{u_2 - u_1}\right]\left[\frac{w_2 - w}{w_2 - w_1}\right] + \bar{p}(u_1,w_2)\left[\frac{u_2 - u}{u_2 - u_1}\right]\left[\frac{w - w_1}{w_2 - w_1}\right]$$

$$+\,\bar{p}(u_2,w_1)\left[\frac{u - u_1}{u_2 - u_1}\right]\left[\frac{w_2 - w}{w_2 - w_1}\right] + \bar{p}(u_2,w_2)\left[\frac{u - u_1}{u_2 - u_1}\right]\left[\frac{w - w_1}{w_2 - w_1}\right]$$

Ruled (or Lofted) Surfaces

Here we specify two of the four boundary curves, $\overline{p}(u,0)$ and $\overline{p}(u,1)$. These two curves can be defined by any of the methods that we discussed (cubic spline, Bezier, B-spline, NURBS, etc.). Points on the surface are obtained by linear interpolation.

$$\overline{p}(u,w) = \underline{\overline{p}(u,0)}(1-w) + \underline{\overline{p}(u,1)}w$$

or

$$\overline{p}(u,w) = \underline{\overline{p}(0,w)}(1-u) + \underline{\overline{p}(1,w)}u$$

If we choose straight lines for the boundaries, the ruled surface becomes a bilinear surface.

An Example

$$\overline{p}(u,w) = \overline{p}(u,0)(1-w) + \overline{p}(u,1)\,w$$

$\overline{p}(u,0)$ and $\overline{p}(u,1)$ are cubic splines with clamped ends

$\boxed{c_1 = \overline{p}(u,0)}$

$p_1 = [0 \quad 0 \quad 0]^T$

$p_2 = [0 \quad 1 \quad 0]^T$

$p_1^{'} = [0 \quad 1 \quad 1]^T$

$p_2^{'} = [0 \quad 1 \quad 1]^T$

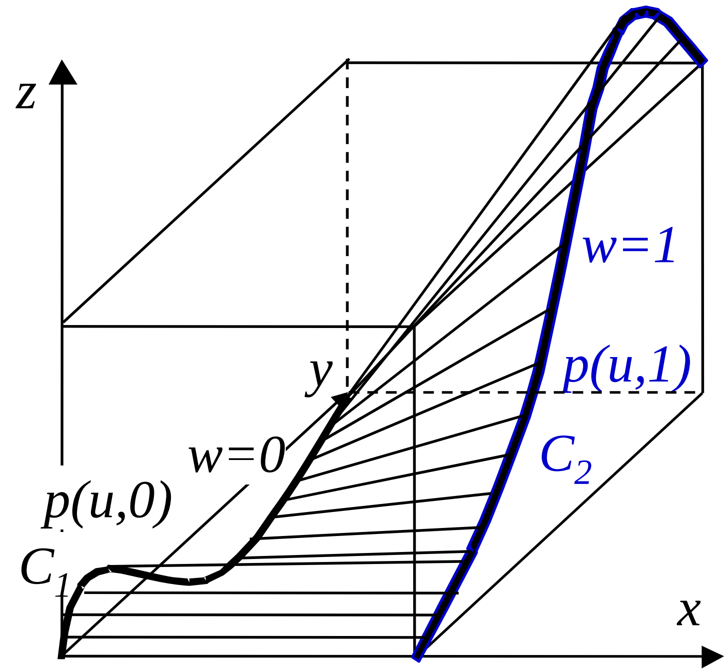
$\boxed{c_2 = \overline{p}(u,1)}$

$p_1 = [1 \quad 0 \quad 0]^T$

$p_2 = [1 \quad 1 \quad 1]^T$

$p_1^{'} = [0 \quad 1 \quad 1]^T$

$p_2^{'} = [0 \quad 1 \quad -1]^T$



For each curve (C_1 and C_2 – *cubic splines*):

$$\vec{P}(u) = (2u^3 - 3u^2 + 1)\vec{P}_0 + (-2u^3 + 3u^2)\vec{P}_1 + (u^3 - 2u^2 + u)\vec{P}_0^{'} + (u^3 - u^2)\vec{P}_1^{'}$$

Substituting,

$$\bar{P}(u,0) = [0\ 0\ 0] + [0\ 1\ 1]u + [0\ 0\ -3]u^2 + [0\ 0\ 2]u^3$$

$$\bar{P}(u,1) = [1\ 0\ 0] + [0\ 1\ 1]u + [0\ 0\ 2]u^2 + [0\ 0\ -2]u^3$$

The equation for the surface is:

$$\bar{P}(u,w) = \bar{P}(u,0)(1-w) + \bar{P}(u,1)w$$

$$= \{[0\ 0\ 0] + [0\ 1\ 1]u + [0\ 0\ -3]u^2 + [0\ 0\ 2]u^3\}$$

$$\times(1-w) + \{[1\ 0\ 0] + [0\ 1\ 1]u$$

$$+ [0\ 0\ 2]u^2 + [0\ 0\ -2]u^3\}w$$

Rearranging in powers of u:

$$\bar{P}(u,w) = [w\ 0\ 0] + [0\ 1\ 1]u + [0\ 0\ (5w-3)]u^2 + [0\ 0\ (2-4w)]u^3$$

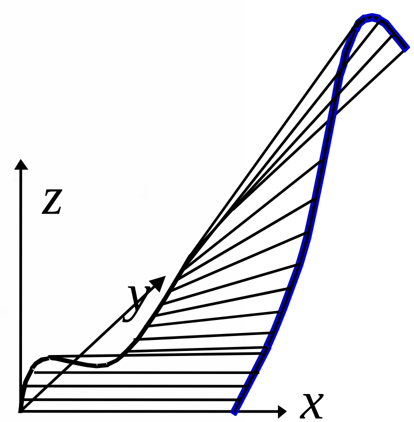
Δ Display and Machining:

Given small Δu, & Δw.

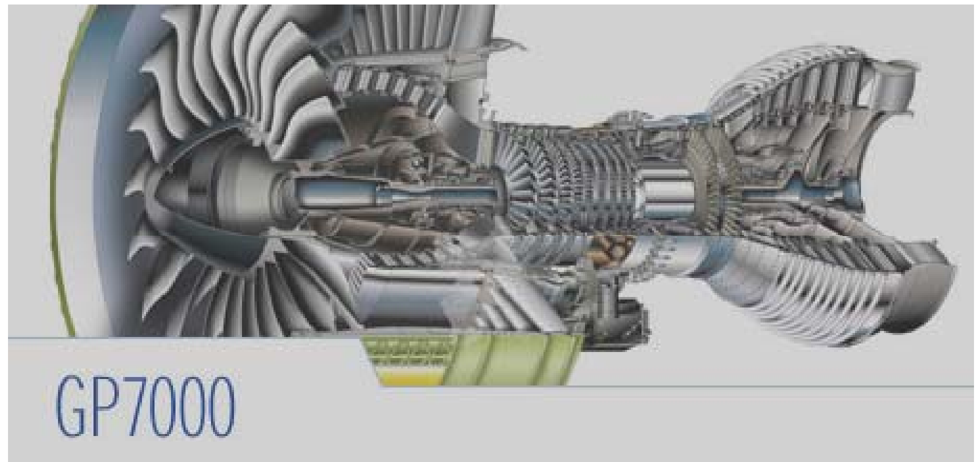
$$u_{i+1} = u_i + \Delta u \qquad u_0 = 0$$

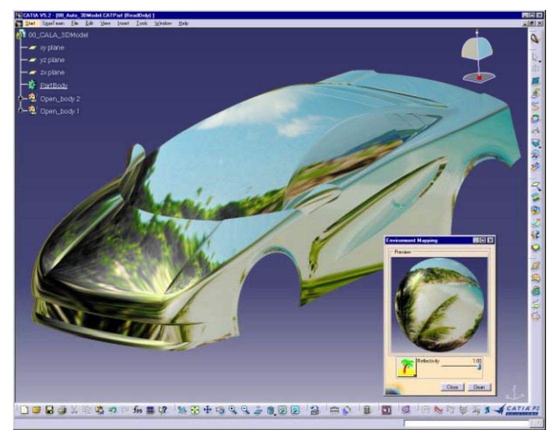
$$w_{i+1} = w_i + \Delta w \qquad w_0 = 0$$

calculate $\bar{P}(u_i, w_i) = \bar{P}_{ij}$ and generate a surface mesh.



Free-form Surface II



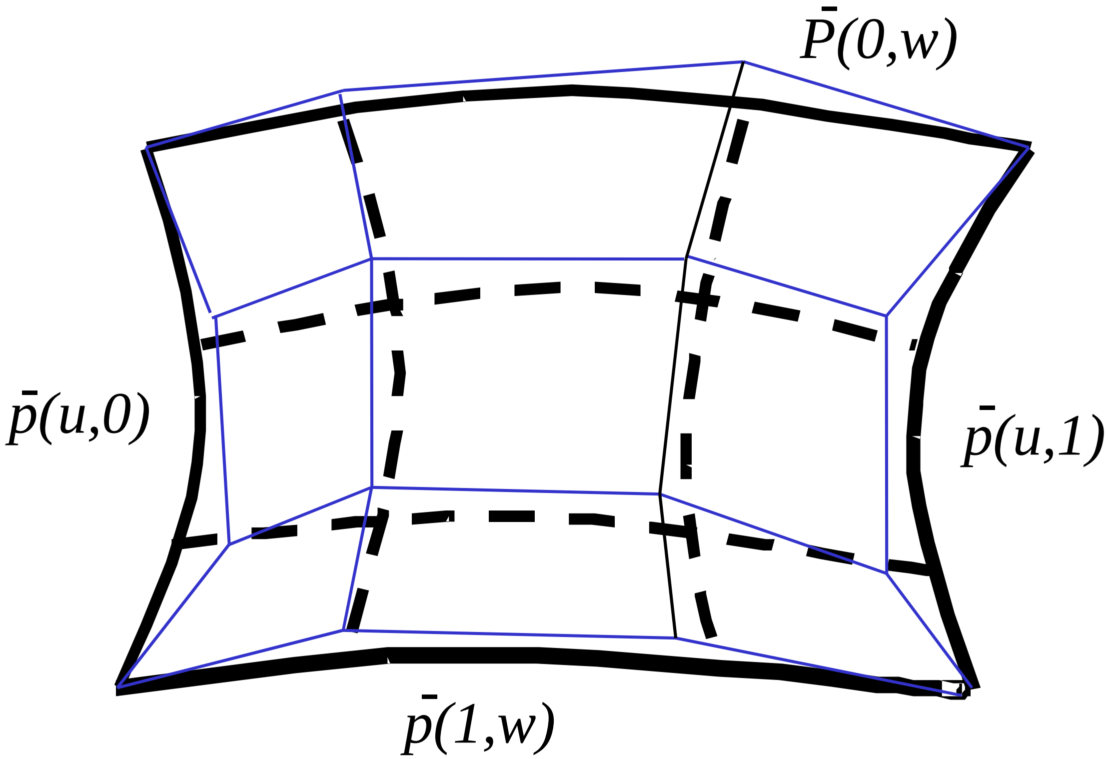


Type of Surfaces

- Planar Surface
- Bilinear Surface
- Ruled (lofted) Surface
- Bezier Surface
- Bi-cubic surface
- B-Spline Surface

Bezier Surface Patch

Bezier surfaces are formed by plotting families of Bezier curves. Changes of control points alter the global shape of the surface patch.



Bezier Surface Patch

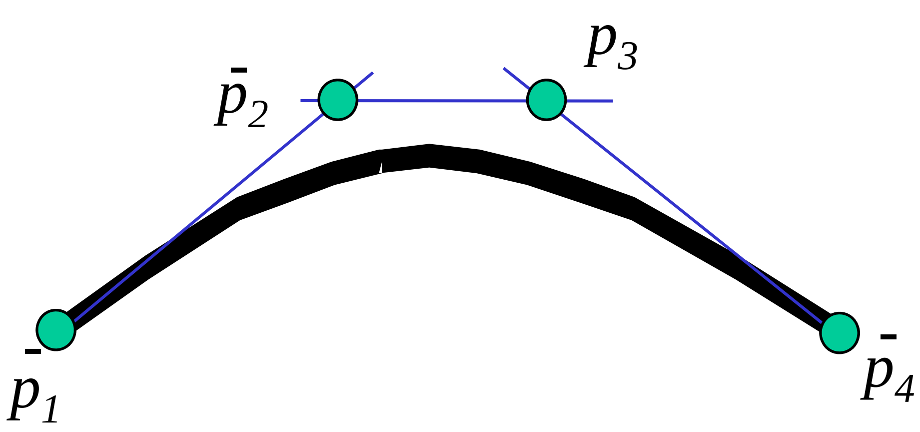
A Bezier Curve

$$\overline{p}(u) = \sum_{i=0}^{n} \overline{p_i} B_{i,n}(u) \qquad 0 \le u \le 1$$

$$B_{i,n}(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i}$$

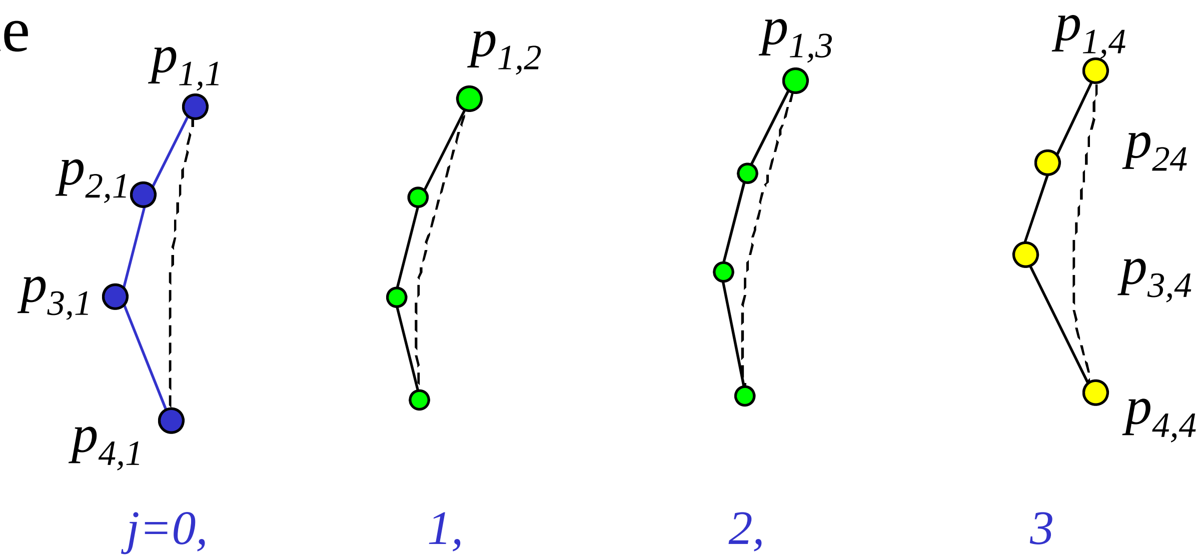
An Example

(4 points)



Bezier Surface Patch

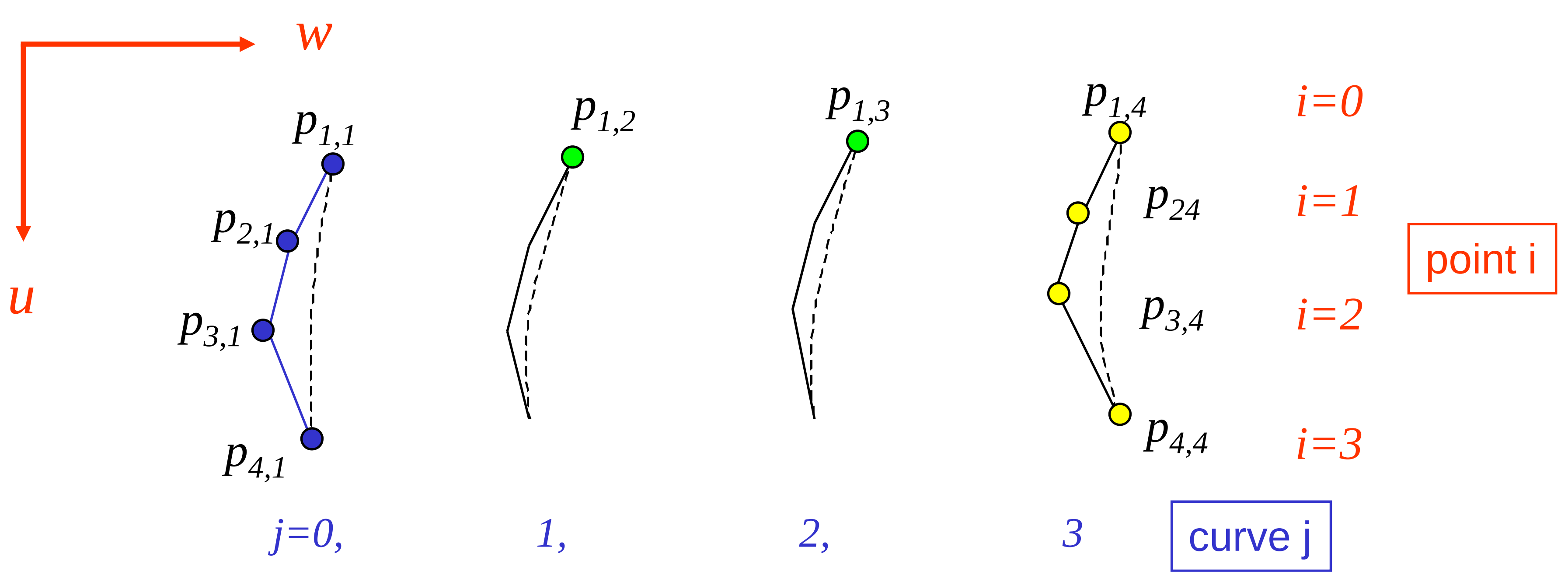
An Example



$$P = \begin{bmatrix} \overline{P_{1,1}} & \overline{P_{1,2}} & \overline{P_{1,3}} & \overline{P_{1,4}} \\ \overline{P_{2,1}} & \overline{P_{2,2}} & \overline{P_{2,3}} & \overline{P_{2,4}} \\ \overline{P_{3,1}} & \overline{P_{3,2}} & \overline{P_{3,3}} & \overline{P_{3,4}} \\ \overline{P_{4,1}} & \overline{P_{4,2}} & \overline{P_{4,3}} & \overline{P_{4,4}} \end{bmatrix} = \left\{ \overline{p_{i+1,\,j+1}} \right\}$$

curve 0 curve 1 curve 2 curve 3

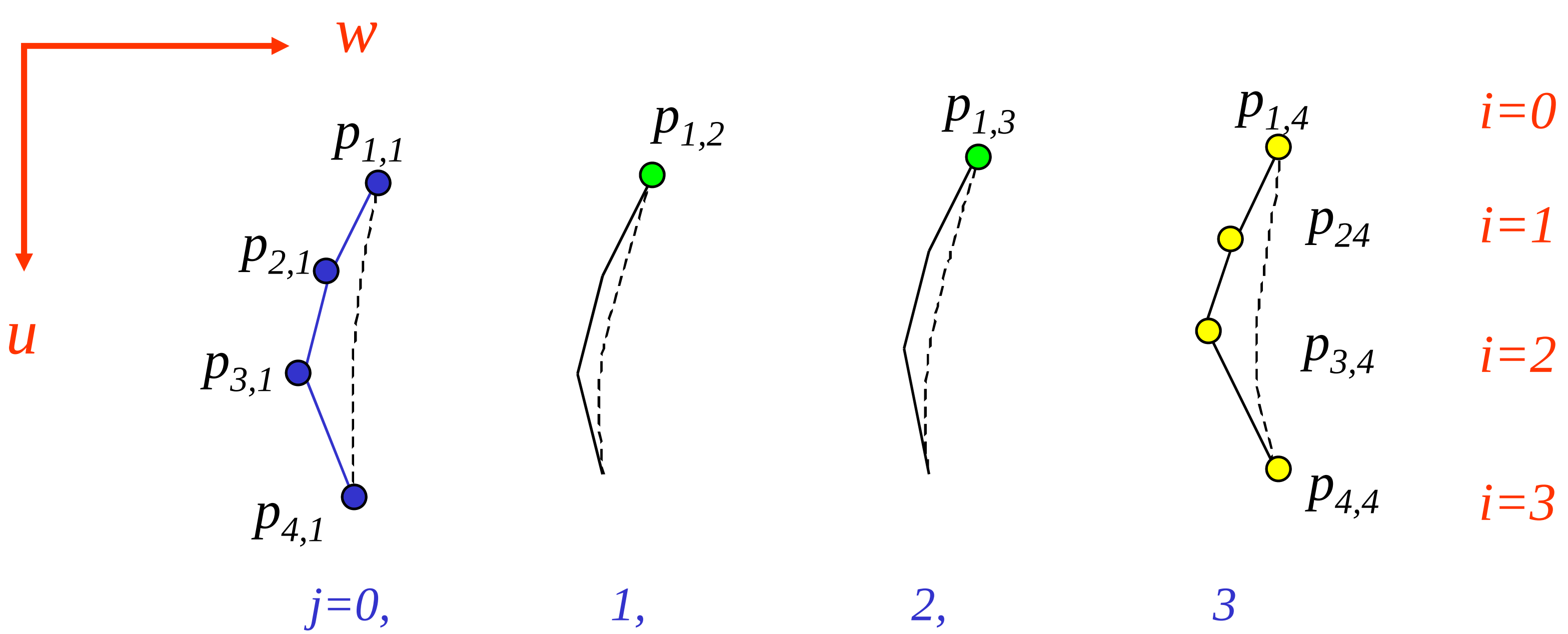
Bezier Surface Patch



$$\overline{p}(u, 0) = \sum_{i=0}^{3} \overline{p_{i+1,1}} * B_{i,3}(u) \qquad (j = 0 \quad or \quad w = 0)$$

$$= (1-u)^3 \overline{p_{1,1}} + 3(1-u)^2 u \overline{p_{2,1}} + 3(1-u) u^2 \overline{p_{3,1}} + u^3 \overline{p_{4,1}}$$

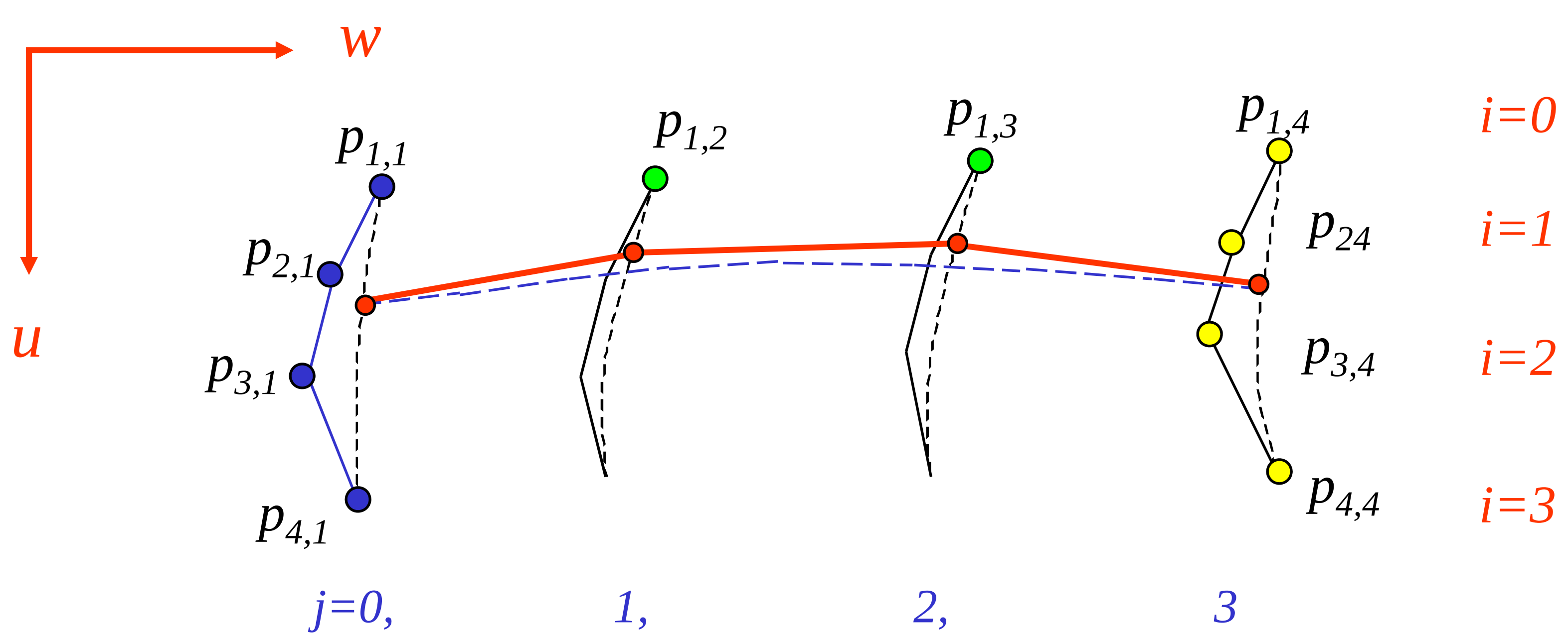
Bezier Surface Patch



For <u>other three</u> Bezier curves:

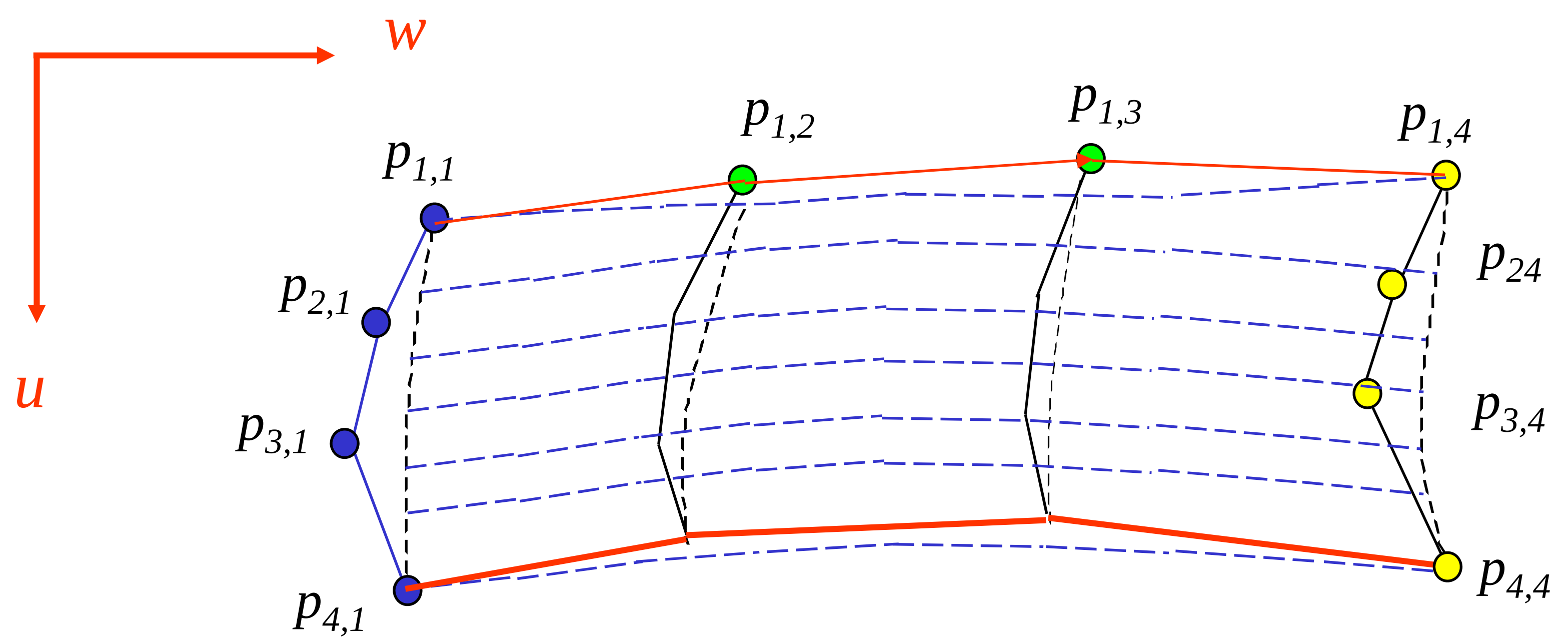
$$\overline{p}\left(u, \frac{j}{n}\right) = \sum_{i=0}^{3} \overline{p_{i+1,\, j+1}} * B_{i,3}(u) \qquad (j = 1,\ 2,\ 3)$$

Bezier Surface Patch



Now we chose a point on each curve ($u=u_o$) to form a new control polygon, and generate a new Bezier curve.

Allow u to continuously change from $u=0$ to $u=1$, the Bezier surface is formed.



w
u
p1,1
p1,2
p1,3
p1,4
p2,1
p24
p3,1
p3,4
p4,1
p4,4

A Cubic Bezier Surface Patch

$$\overline{p}(u,w)=\sum_{j=0}^{3}\overline{p}\left(u,\frac{j}{n}\right)\times B_{j,3}(w) \qquad \begin{cases}0\le u\le 1\\ 0\le w\le 1\end{cases}$$

$$\boxed{=\sum_{j=0}^{3}\sum_{i=0}^{3}\overline{p_{i+1,j+1}}\times B_{i,3}(u)\times B_{j,3}(w)}$$

$$=\begin{bmatrix}(1-u)^3 & 3(1-u)^2u & 3(1-u)u^2 & u^3\end{bmatrix}\begin{bmatrix}P_{11} & P_{12} & P_{13} & P_{14}\\ P_{21} & P_{22} & P_{23} & P_{24}\\ P_{31} & P_{32} & P_{33} & P_{34}\\ P_{41} & P_{42} & P_{43} & P_{44}\end{bmatrix}\begin{bmatrix}(1-w)^3\\ 3(1-w)^2w\\ 3(1-w)w^2\\ w^3\end{bmatrix}$$

$$\boxed{=U^T[M_B][P][M_B]^T W}$$

$$[M_B]=\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

A General Bezier Surface Patch

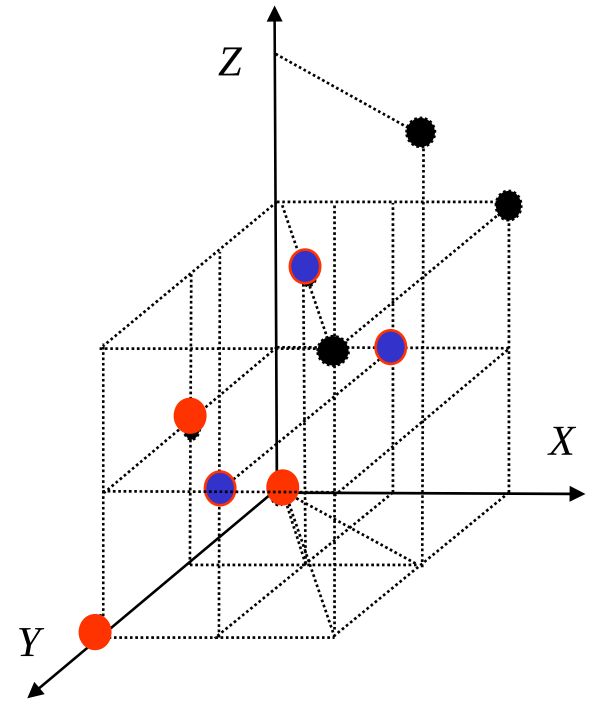
A general $n \times m$ Bezier surface:

$$\boxed{\overline{p}(u,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} \overline{p_{i+1,j+1}} B_{i,n}(u) B_{j,m}(w)}$$

$$B_{i,n}(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i}$$

$$B_{j,m}(u) = \frac{m!}{j!(m-j)!} w^j (1-w)^{m-j}$$

An Example



A Bezier surface patch is specified by 9 control points:

$\boldsymbol{p}_{00} = (0,\ 0,\ 0)^T;\ \ \boldsymbol{p}_{01} = (0,\ 1,\ 1)^T;\ \ \boldsymbol{p}_{02} = (0,\ 2,\ 0)^T$

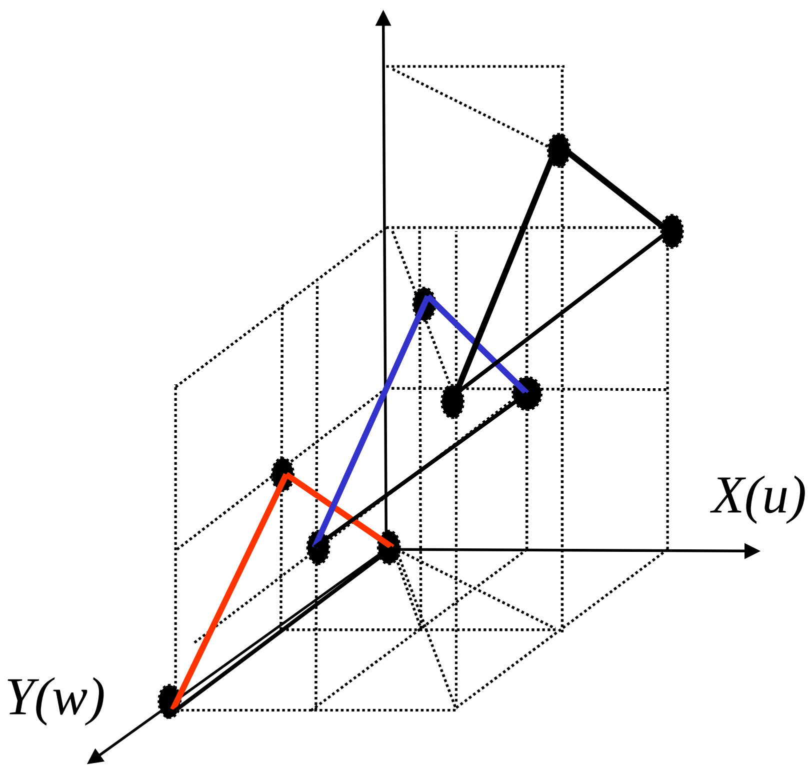
$\boldsymbol{p}_{10} = (1,\ 0,\ 1)^T;\ \ \boldsymbol{p}_{11} = (1,\ 1,\ 2)^T;\ \ \boldsymbol{p}_{12} = (1,\ 2,\ 1)^T$

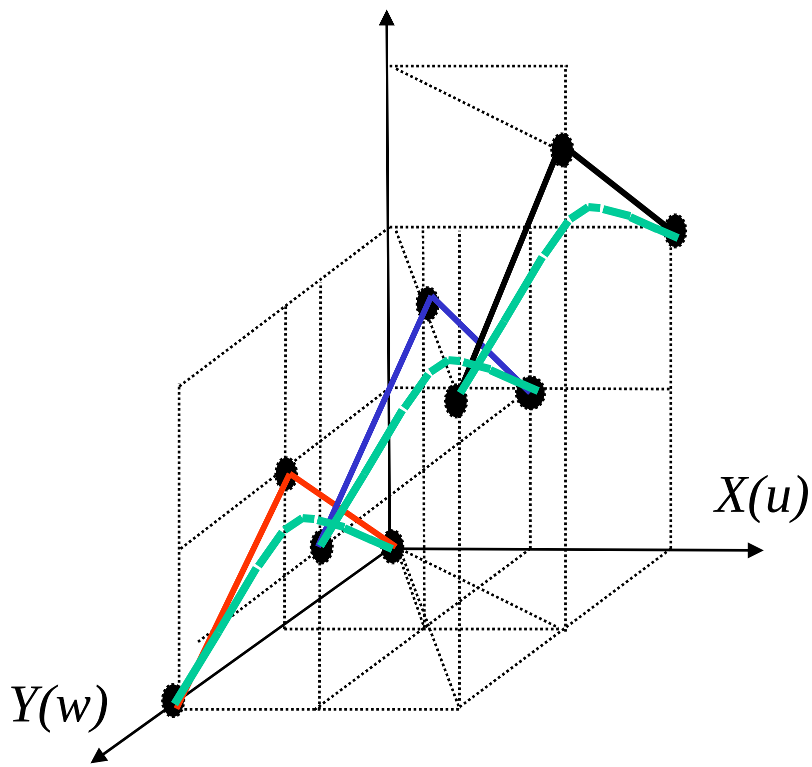
$\boldsymbol{p}_{20} = (2,\ 0,\ 2)^T;\ \ \boldsymbol{p}_{21} = (2,\ 1,\ 3)^T;\ \ \boldsymbol{p}_{22} = (2,\ 2,\ 2)^T$

1. Plot the control polygon and sketch the surface patch of $\boldsymbol{p}(u,w)$.
2. Given $\boldsymbol{p}_{00}\boldsymbol{p}_{01}\boldsymbol{p}_{02}$ as the $u = 0$ curve and $\boldsymbol{p}_{00}\boldsymbol{p}_{10}\boldsymbol{p}_{20}$ as the $w = 0$ curve, derive the Beizer curve expression for the two boundary curves, $\boldsymbol{q}(0,w)$ and $\boldsymbol{q}(u,0)$.
3. Derive the mathematical representation of the surface patch $\boldsymbol{p}(u,w)$.
4. Calculate $\boldsymbol{p}(0.5,\ 0.5)$

Solution

1. Sketch





2. If $\mathbf{p}_{00}\mathbf{p}_{01}\mathbf{p}_{02}$ defines the u=0 curve,

Solution

$$\vec{p}(u) = \sum_{i=0}^{n} \vec{p}_i B_{i,n}(u)$$

$$B_{i,n}(u) = \frac{n!}{i!(n-i)\ !} u^i (1-u)^{n-i}$$

$$\vec{p}(u) = \underline{(1-u)^2}\,\vec{p_0} + \underline{2u(1-u)}\,\vec{p_1} + \underline{u^2}\,\vec{p_2}$$

$$P(0,w) = \underline{(1-w)}^2 P_{00} + \underline{2w(1-w)}P_{01} + \underline{w}^2 P_{02}$$

$$= [0 \quad 2w \quad 2w(1-w)]^T$$

If $\mathbf{p}_{00}\mathbf{p}_{10}\mathbf{p}_{20}$ defines the $w = 0$ curve, similarly,

$$P(u,0) = \underline{(1-u)}^2 P_{00} + \underline{2u(1-u)}P_{10} + \underline{u}^2 P_{20}$$

$$= [2u \quad 0 \quad 2u)]^T$$

Solution

3. Surface Patch

$$P(u,w)=\sum_{j=0}^{2}\sum_{i=0}^{2}P_{i,j}*B_{i,2}(u)*B_{j,2}(w) \qquad \begin{cases}0\le u\le 1\\ 0\le w\le 1\end{cases}$$

$$=\left[(1-u)^2 \quad 2u(1-u) \quad u^2\right]\begin{bmatrix}P_{00} & P_{01} & P_{02}\\ P_{10} & P_{11} & P_{12}\\ P_{20} & P_{21} & P_{22}\end{bmatrix}\begin{bmatrix}(1-w)^2\\ 2w(1-w)\\ w^2\end{bmatrix}$$

$$\begin{aligned}P(u,w)=&(1-u)^2(1-w)^2P_{00}+2u(1-u)(1-w)^2P_{10}+u^2(1-w)^2P_{20}\\ &+2(1-u)^2w(1-w)P_{01}+2u(1-u)\cdot 2w(1-w)P_{11}+u^2\cdot 2w(1-w)P_{21}\\ &+(1-u)^2w^2P_{02}+2u(1-u)w^2P_{12}+u^2w^2P_{22}\end{aligned}$$

Solution

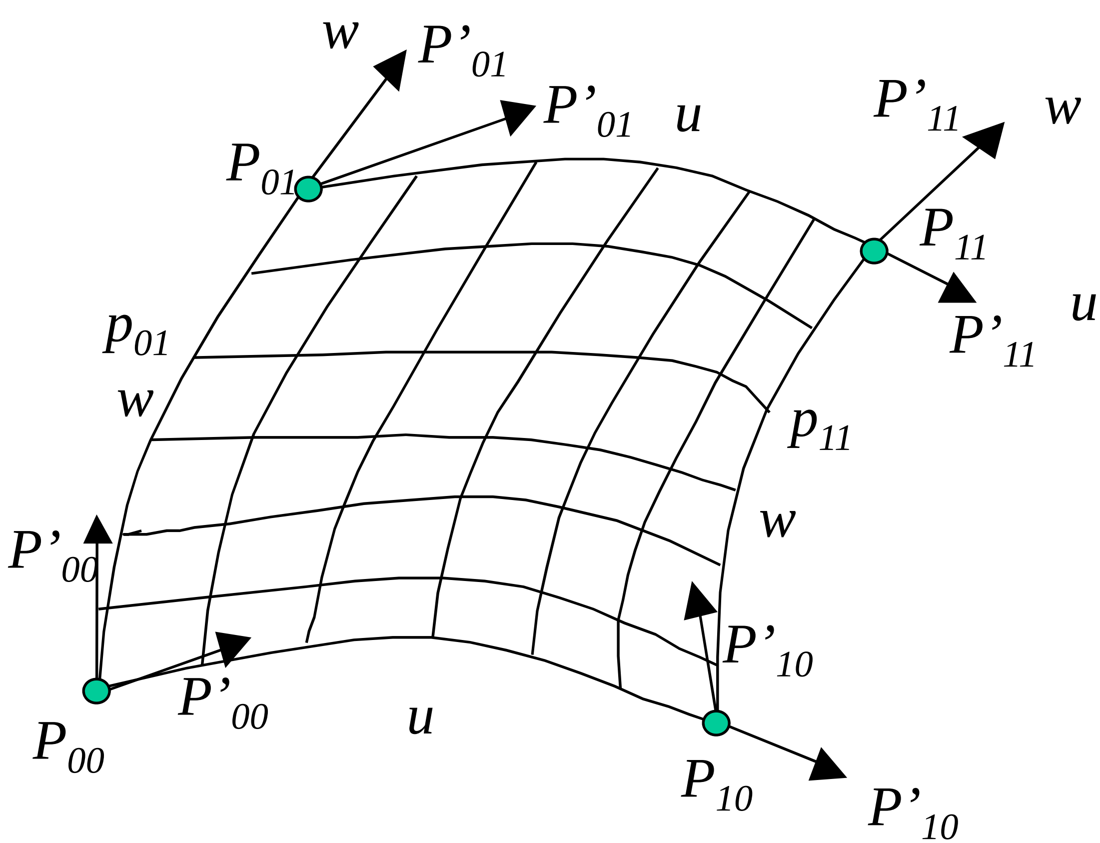
3. Calculate $P(0.5,\ 0.5)$

$$P(u,w) = (1-u)^2(1-w)^2 P_{00} + 2u(1-u)(1-w)^2 P_{10} + u^2(1-w)^2 P_{20}$$
$$+ 2(1-u)^2 w(1-w) P_{01} + 2u(1-u)\cdot 2w(1-w) P_{11} + u^2 \cdot 2w(1-w) P_{21}$$
$$+ (1-u)^2 w^2 P_{02} + 2u(1-u) w^2 P_{12} + u^2 w^2 P_{22}$$

$P(0.5,\ 0.5) = [\ 1 \quad 1 \quad 1.5]^T$

$X \quad Y \quad Z$

Bi-Cubic Surface Patch



Bi-Cubic Surface Patch

A Cubic Spline

$$\vec{p}(u) = \sum_{i=0}^{3} \vec{C}_i u^i \qquad (0 \le u \le 1)$$

Unknowns:
  4x3 coefficients
Needed:
  2 end points (2x3)
  2 end slopes (2x3)

A Bi-cubic Surface Patch

$$\overline{p}(u,w) = \left[x(u,w), \quad y(u,w), \quad z(u,w) \right]^T$$

$$= \sum_{i=0}^{3} \sum_{j=0}^{3} \overline{a_{ij}} u^i w^j$$

Unknowns:
  16x3 coefficients
Needed:
  4 corner points (4x3)
  4x2 end slopes (8x3)
  4 twist vectors (4x3)

Bi-Cubic Surface Patch

$$\overline{p}(u,w)=[x(u,w),\quad y(u,w),\quad z(u,w)]^T=\sum_{i=0}^{3}\sum_{j=0}^{3}\overline{a_{ij}}u^i w^j$$

$$\overline{p}(u,w)=UAW^T=\begin{bmatrix}u^3 & u^2 & u & 1\end{bmatrix}\begin{bmatrix}\overline{a_{33}} & \overline{a_{32}} & \overline{a_{31}} & \overline{a_{30}}\\ \overline{a_{23}} & \overline{a_{22}} & \overline{a_{21}} & \overline{a_{20}}\\ \overline{a_{13}} & \overline{a_{12}} & \overline{a_{11}} & \overline{a_{10}}\\ \overline{a_{03}} & \overline{a_{02}} & \overline{a_{01}} & \overline{a_{00}}\end{bmatrix}\begin{bmatrix}w^3\\ w^2\\ w\\ 1\end{bmatrix}$$

$$\boxed{A=M_H B M_H{}^T}$$

$$B=\begin{bmatrix}[P] & [P_w]\\ [P_u] & [P_{uw}]\end{bmatrix}=\begin{bmatrix}\overline{p_{00}} & \overline{p_{01}} & \overline{p_{00}}^{w} & \overline{p_{01}}^{w}\\ \overline{p_{10}} & \overline{p_{11}} & \overline{p_{10}}^{w} & \overline{p_{11}}^{w}\\ \overline{p_{00}}^{u} & \overline{p_{01}}^{u} & \overline{p_{00}}^{uw} & \overline{p_{01}}^{uw}\\ \overline{p_{10}}^{u} & \overline{p_{11}}^{u} & \overline{p_{10}}^{uw} & \overline{p_{11}}^{uw}\end{bmatrix}$$

$$M_H=\begin{bmatrix}2 & -2 & 1 & 1\\ -3 & 3 & -2 & -1\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

Bi-Cubic Surface Patch

$$A = M_H B M_H{}^T$$

$$B = \begin{bmatrix} [P] & [P_w] \\ [P_u] & [P_{uw}] \end{bmatrix} = \begin{bmatrix} \overline{p_{00}} & \overline{p_{01}} & \overline{p_{00}}^{w} & \overline{p_{01}}^{w} \\ \overline{p_{10}} & \overline{p_{11}} & \overline{p_{10}}^{w} & \overline{p_{11}}^{w} \\ \overline{p_{00}}^{u} & \overline{p_{01}}^{u} & \overline{p_{00}}^{uw} & \overline{p_{01}}^{uw} \\ \overline{p_{10}}^{u} & \overline{p_{11}}^{u} & \overline{p_{10}}^{uw} & \overline{p_{11}}^{uw} \end{bmatrix}$$

$$M_H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

A Closer Look

4 corners

$$B=\begin{bmatrix}[P] & [P_w]\\ [P_u] & [P_{uw}]\end{bmatrix}=\begin{bmatrix}\bar{p}_{00} & \bar{p}_{01} & \bar{p}^{w}_{00} & \bar{p}^{w}_{01}\\ \bar{p}_{10} & \bar{p}_{11} & \bar{p}^{w}_{10} & \bar{p}^{w}_{11}\\ \bar{p}^{u}_{00} & \bar{p}^{u}_{01} & \bar{p}^{uw}_{00} & \bar{p}^{uw}_{01}\\ \bar{p}^{u}_{10} & \bar{p}^{u}_{11} & \bar{p}^{uw}_{10} & \bar{p}^{uw}_{11}\end{bmatrix}$$

u=0 curve

u=1 curve

twist vectors

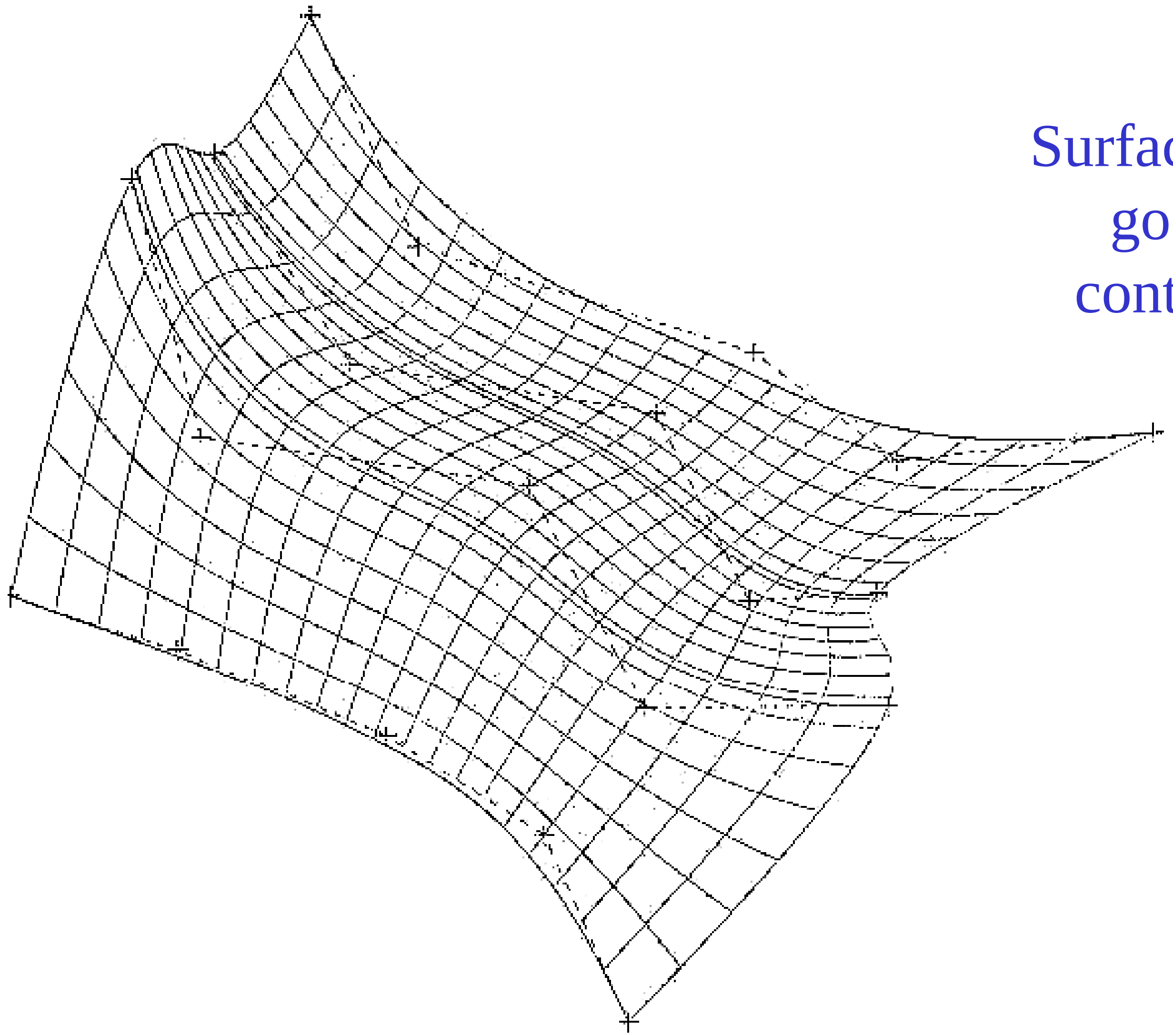
w=0 curve

w=1 curve

Notes for Bi-cubic Surface Patch

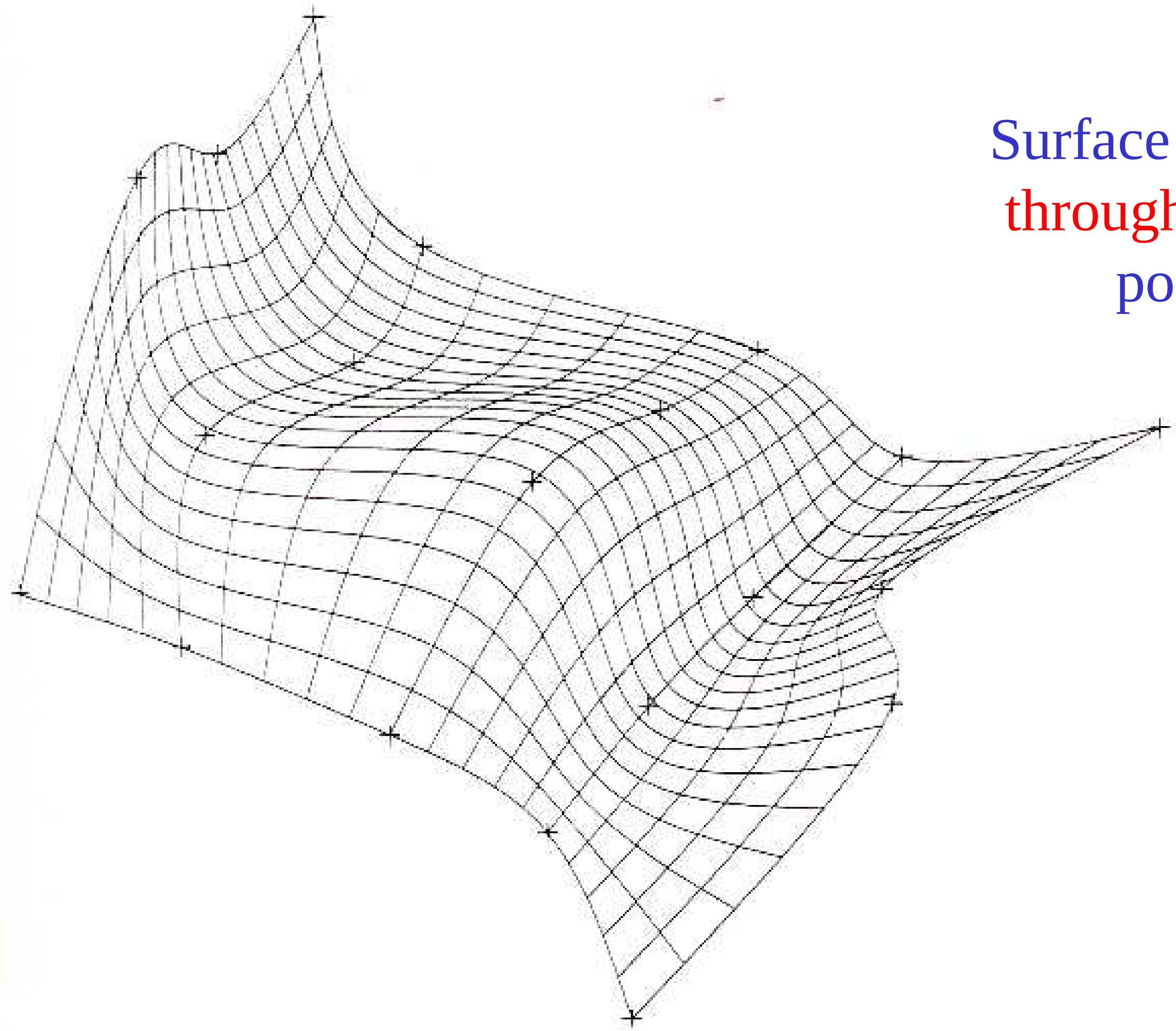
- The surface patch is determined by 4 boundaries. Use $u, w = 0, 1$ to generate the boundary (interactively) of the surface.
- The four boundaries are cubic splines.
- Characteristics of the bi-cubic surface patch are very similar to those of the cubic spline, namely, lack of local control and the order of curve is fixed.
- Requirement of tangent and twist vectors as input data doesn't fit very well the design environment

B-Spline Surface



Surface patch not go through control points

B-Spline Surface



Surface patch go through control points

B-Spline Surface

$$P(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{ij} \underline{N_{i,k}(u) N_{j,l}(v)}, \quad 0 \le u \le u_{\max}, \ \ 0 \le v \le v_{\max}$$

$$= [N_{0,k}(u) \ \ N_{1,k}(u) \ \ \cdots \ \ N_{n,k}(u)] \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0m} \\ P_{10} & P_{11} & \cdots & P_{1m} \\ \vdots & \vdots & & \vdots \\ P_{n0} & P_{n1} & \cdots & P_{nm} \end{bmatrix} \begin{bmatrix} N_{0,l}(v) \\ N_{1,l}(v) \\ \vdots \\ N_{m,l}(v) \end{bmatrix}$$

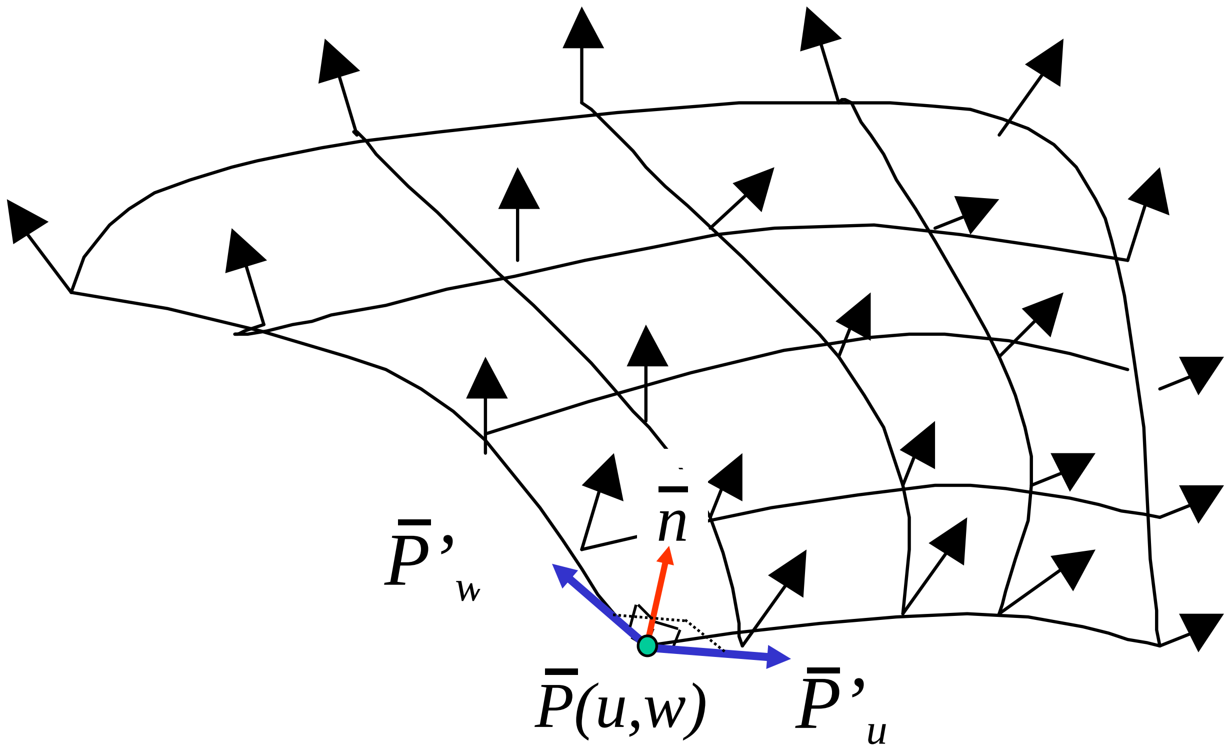
Features: local control and variation of degree

Surface Manipulations

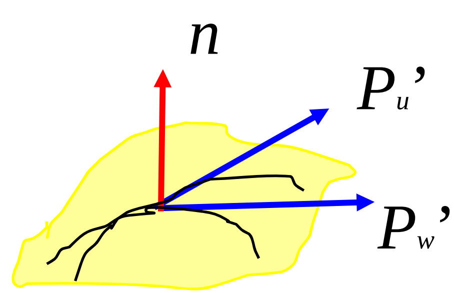
- Offset
- Blend
- Display
- Segmentation (division)
- Trimming
- Intersection
- Projection
- Transformation

Surface Normal

Applications: Direction, Distance Calculation, and Machining



Surface Offset

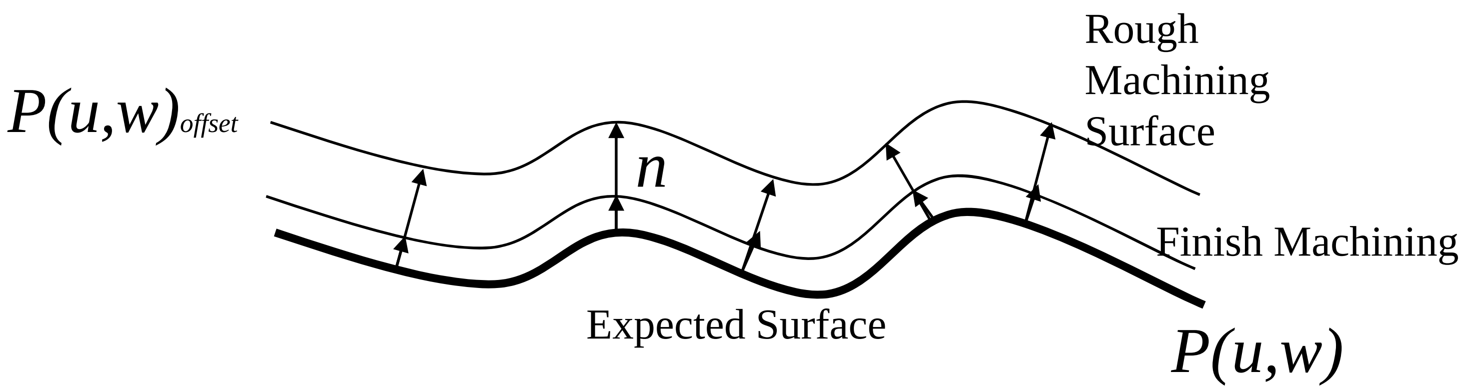


$$\boxed{\overline{n}(u,w) = \overline{p_u^{'}}(u,w) \times \overline{p_w^{'}}(u,w)}$$

$$\overline{p_u^{'}}(u,w) = \frac{\partial \overline{p}(u,w)}{\partial u}; \overline{p_w^{'}}(u,w) = \frac{\partial \overline{p}(u,w)}{\partial w}$$

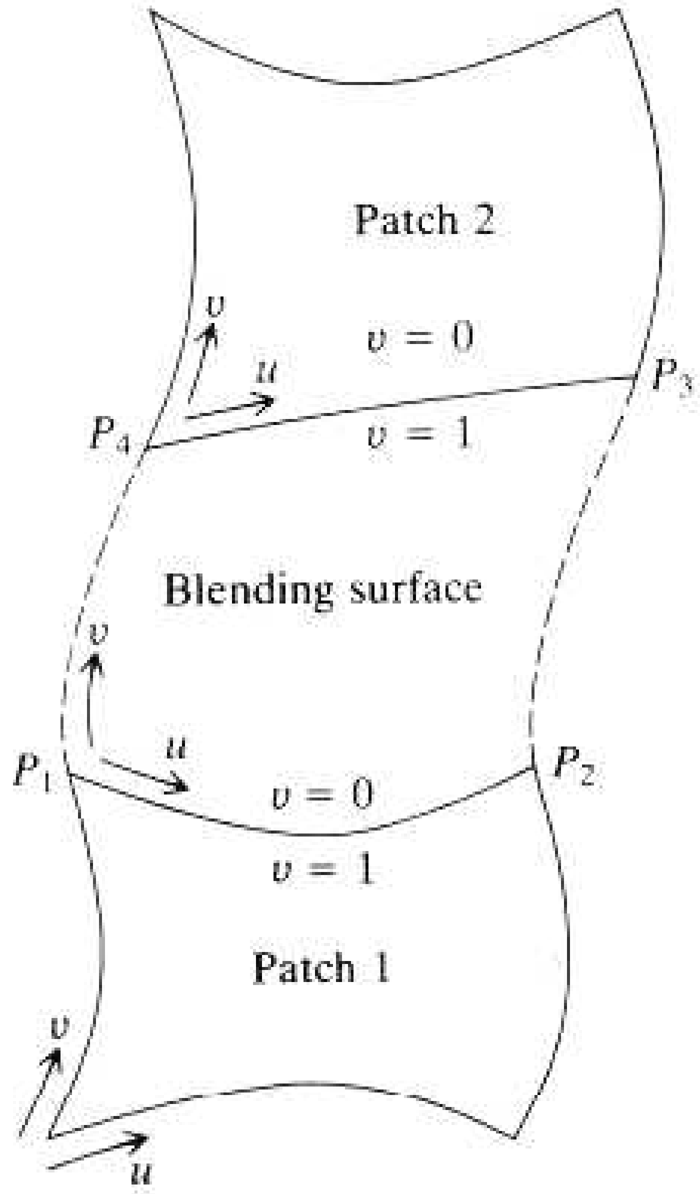
Generating the offset surface of a curved surface:

$$\boxed{\overline{p}(u,w)_{offset} = \overline{p}(u,w) + \overline{n}(u,w) * d(u,w)}$$



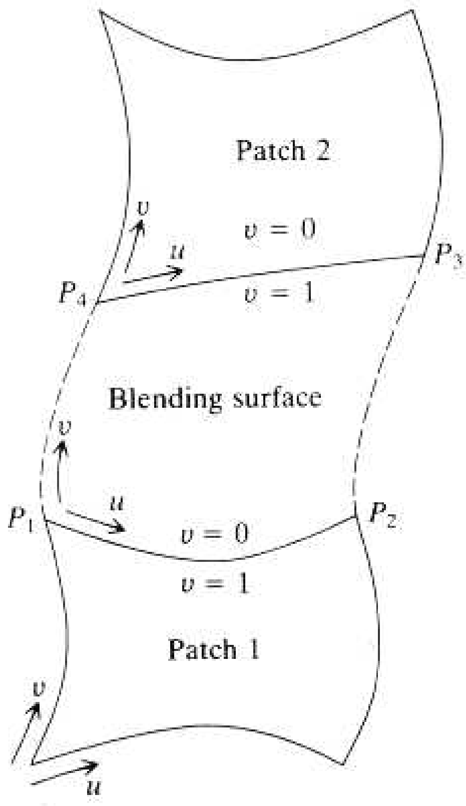
Blending Surfaces

A blending surface is a surface that connects two adjacent surfaces or patches. A blending surface is usually created for two given surface patches.



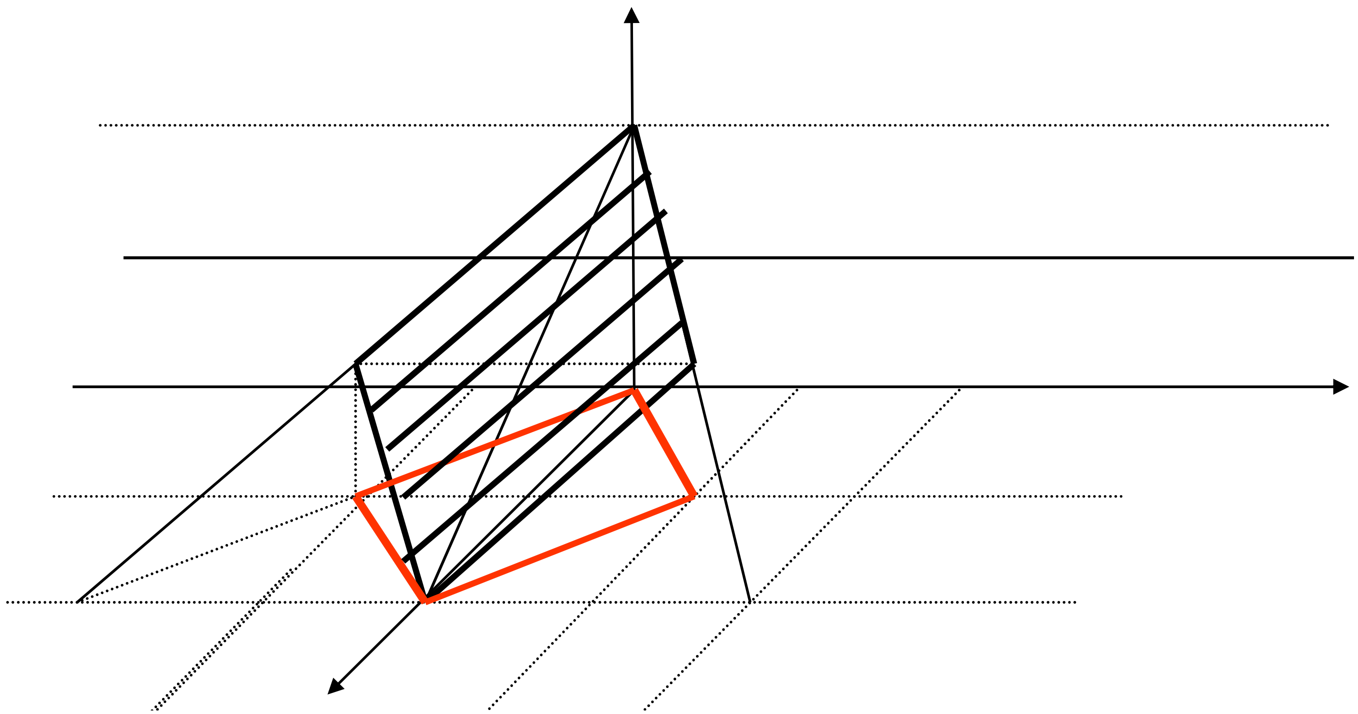
Blending Surfaces

A bi-cubic blending surface can be readily generated



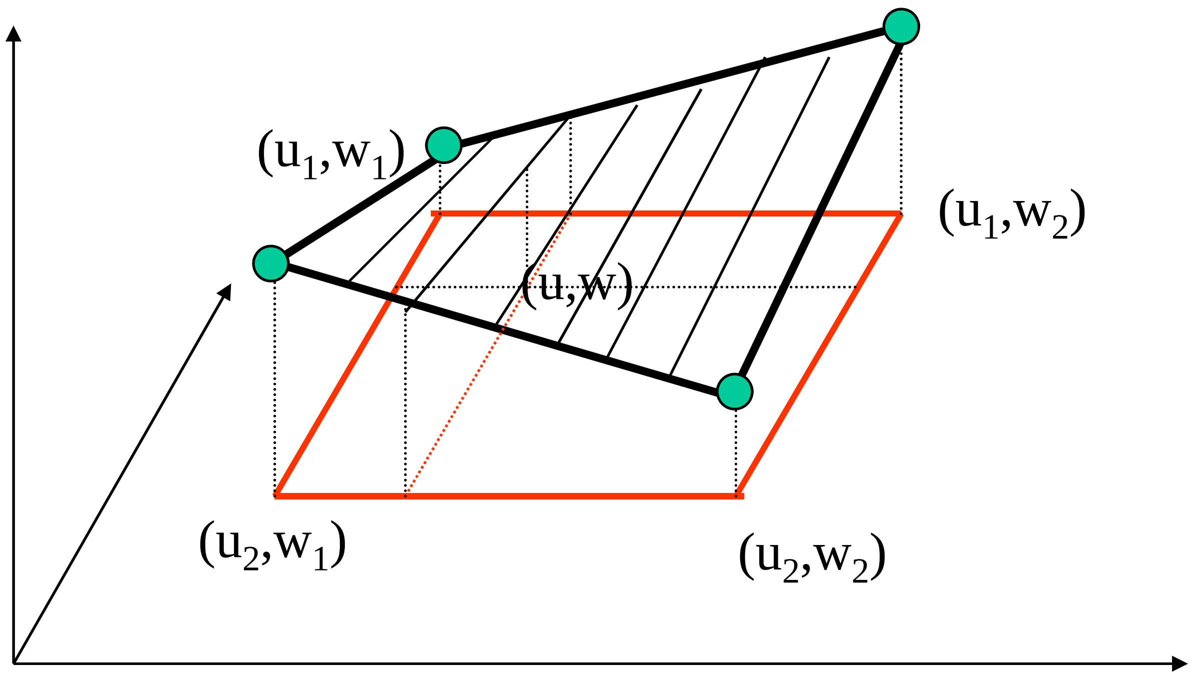
How about Bezier or B-Spline blending surfaces?

Review: A Plane (Patch)



$$
\begin{cases}
x = x(u,w) \\
y = y(u,w) \\
z = z(u,w) = -\dfrac{D}{c} - \dfrac{B}{c}\, y(u,w) - \dfrac{A}{c}\, x(u,w)
\end{cases}
\qquad Ax + By + Cz + D = 0
$$

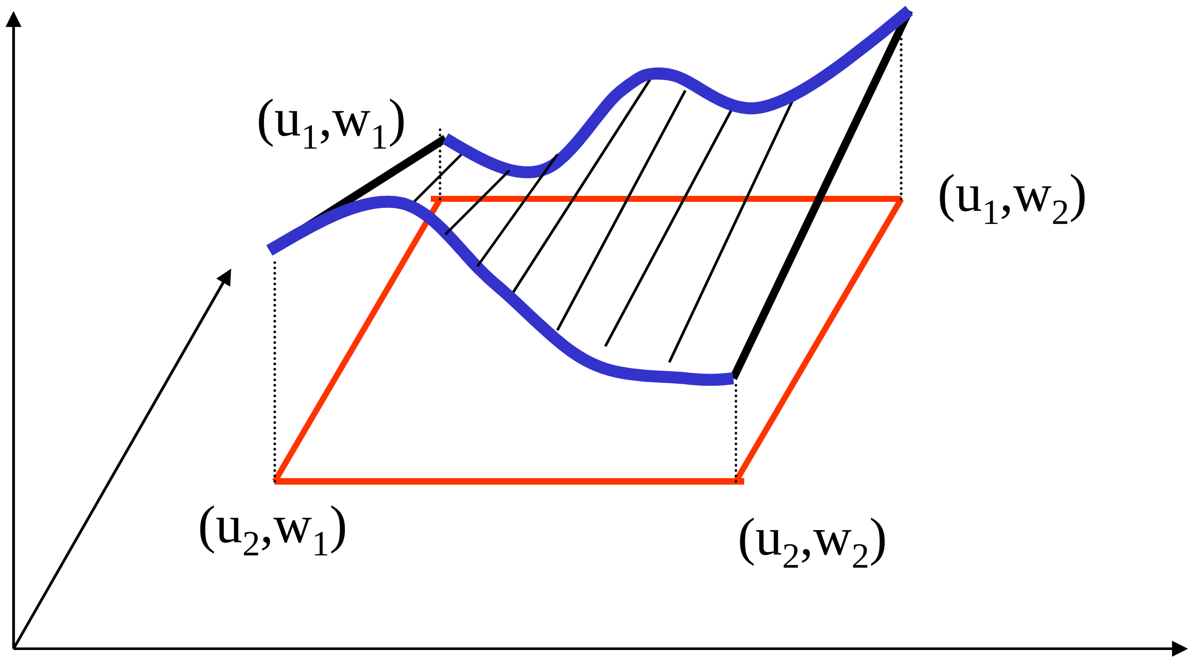
Review: Bilinear Surface



$$\vec{P}(u,w) = [1-u \quad u]\begin{bmatrix} \vec{p}(u_1,w_1) & \vec{p}(u_1,w_2) \\ \vec{p}(u_2,w_1) & \vec{p}(u_2,w_2) \end{bmatrix}\begin{bmatrix} 1-w \\ w \end{bmatrix}$$

$$u,w \in [0,1]$$

Review: Ruled (lofted) Surface

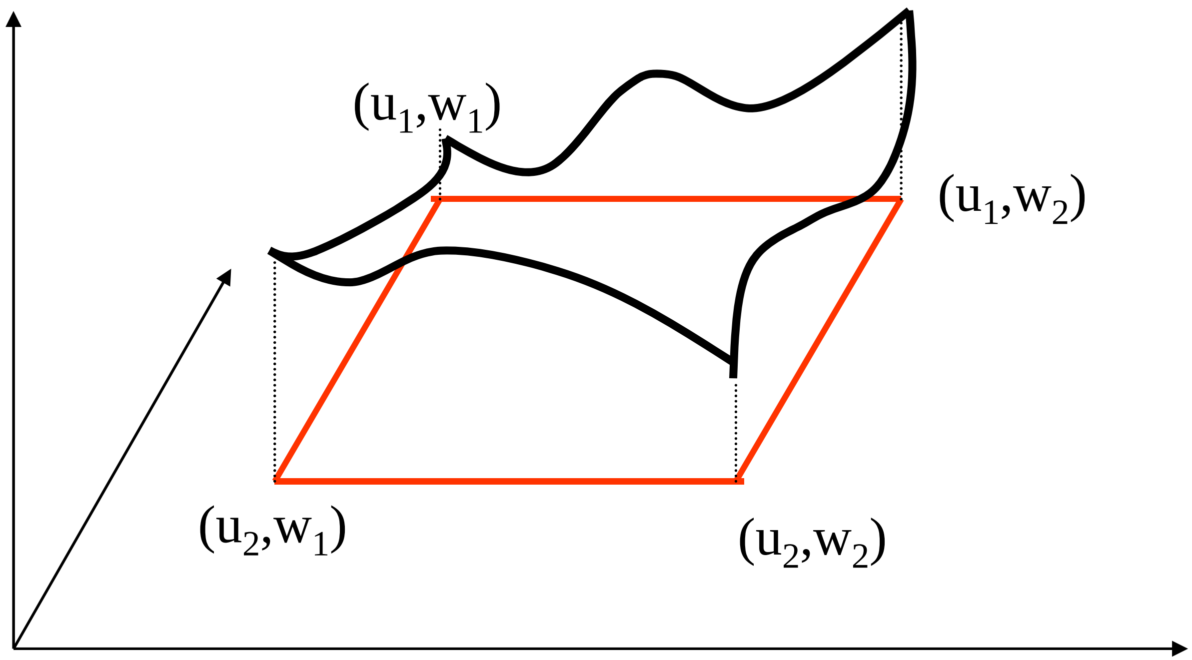


$$\vec{P}(u,w) = \vec{p}(u,0)(1-w) + \vec{p}(u,1)w$$

$$u, w \in [0,1]$$

$$or\ \ \vec{P}(u,w) = \vec{p}(0,w)(1-u) + \vec{p}(1,w)u$$

Review: Bezier Surface Patch



$$\vec{P}(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}\vec{p}_{i+1,\,j+1}B_{i,n}(u)B_{j,m}(w)\qquad u,w\in[0,1]$$

Quick Questions

1. What are the parameters required to define a planar, bilinear, lofted/ruled, and Bezier surface patch?
2. What are the differences between a bilinear surface and a ruled surface?
3. Can we increase the order of a bi-cubic surface patch by introducing more points?
4. Bezier surface, in contrast to Bezier curve, can be locally tuned if appropriate mathematical manipulation is applied.
5. Is the blending surface usually a bi-cubic surface, Bezier surface or B-spline surface? Why?

Summary

- **Planar, bilinear, and ruled surfaces**
- Cubic, **Bezier**, B-Spline Surfaces

  - Properties similar to corresponding curves

  - Curves are rudimental for surfaces

  - An extension to two dimension (u, w)
- Surface manipulation (offset, and blending)