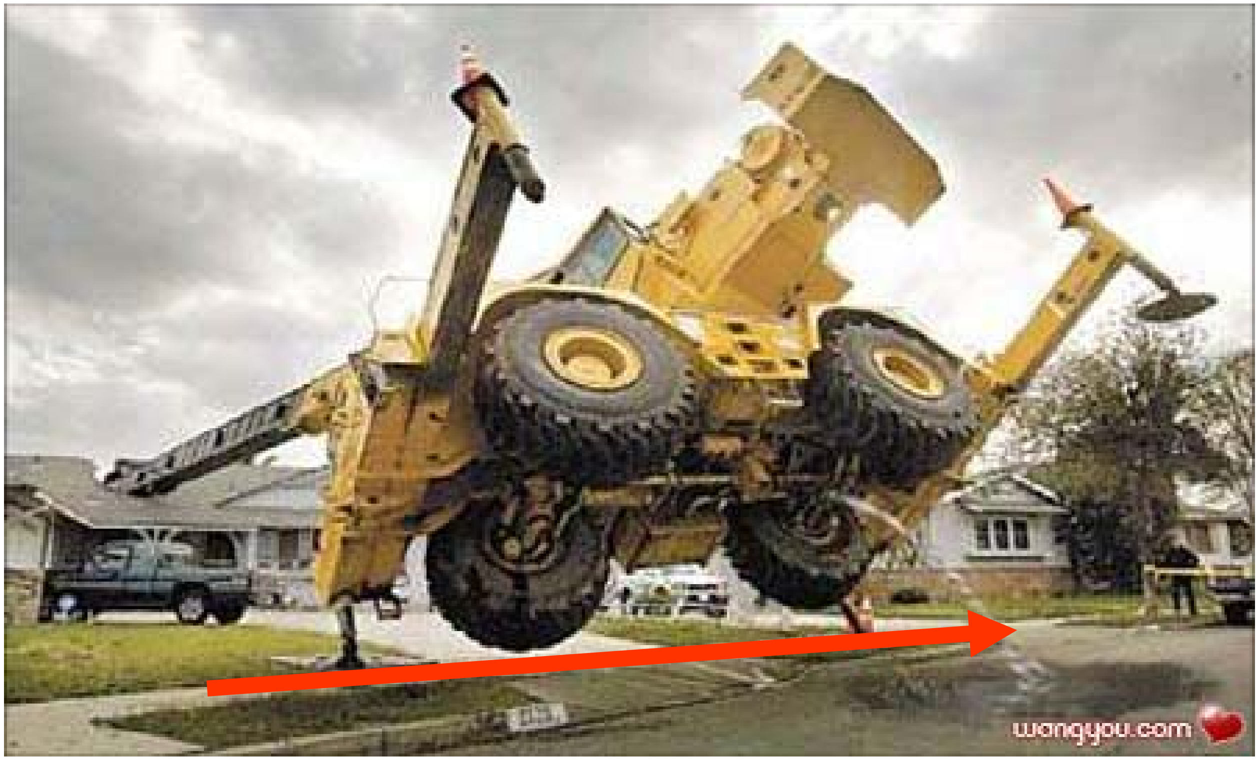
Rotation about an Arbitrary Axis (Line)



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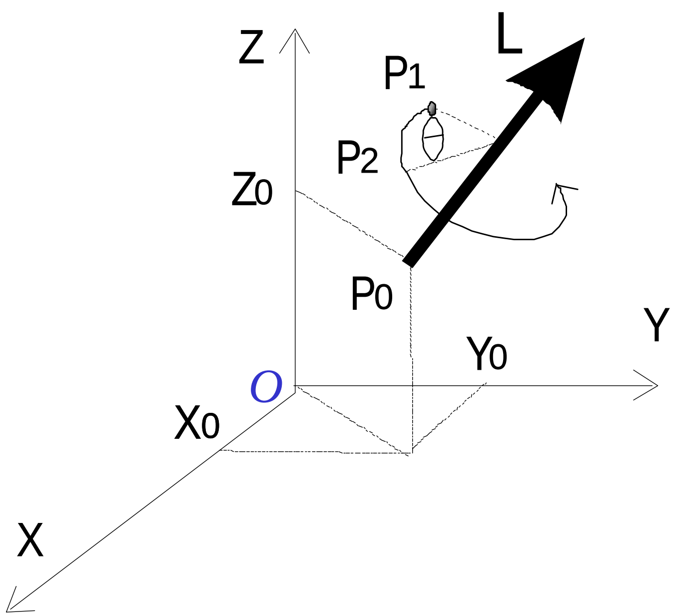
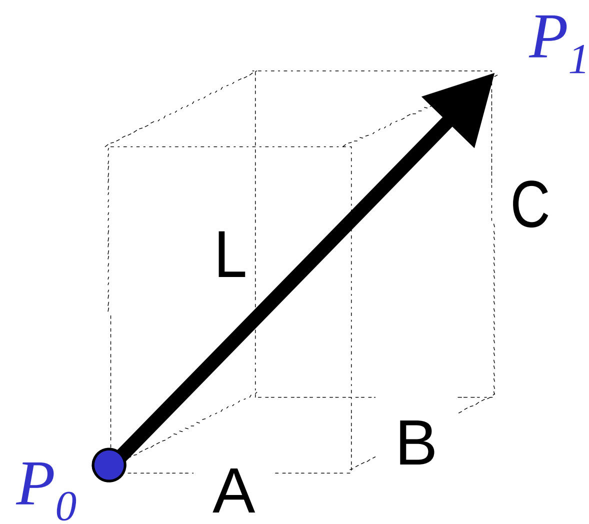




$$x = Au + x_0$$

$$y = Bu + y_0 \qquad 0 <= u <= 1$$

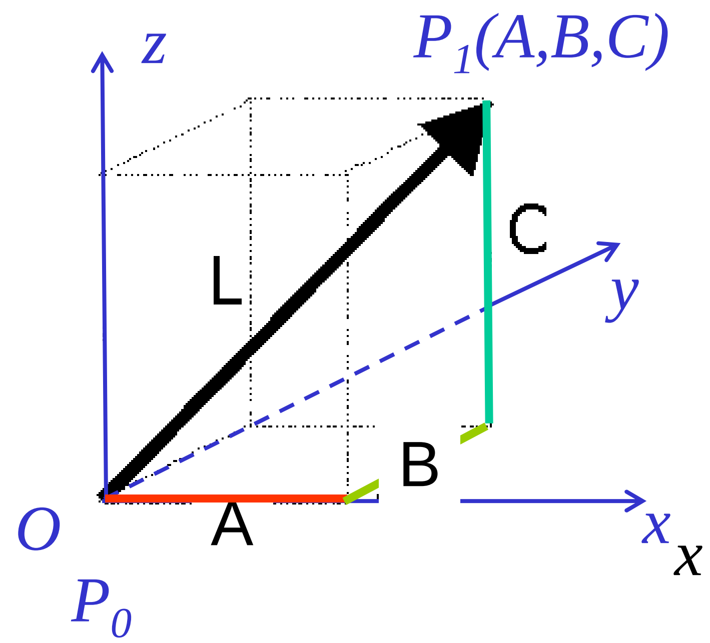
$$z = Cu + z_0$$

$$L = \sqrt{A^2 + B^2 + C^2}\, u$$

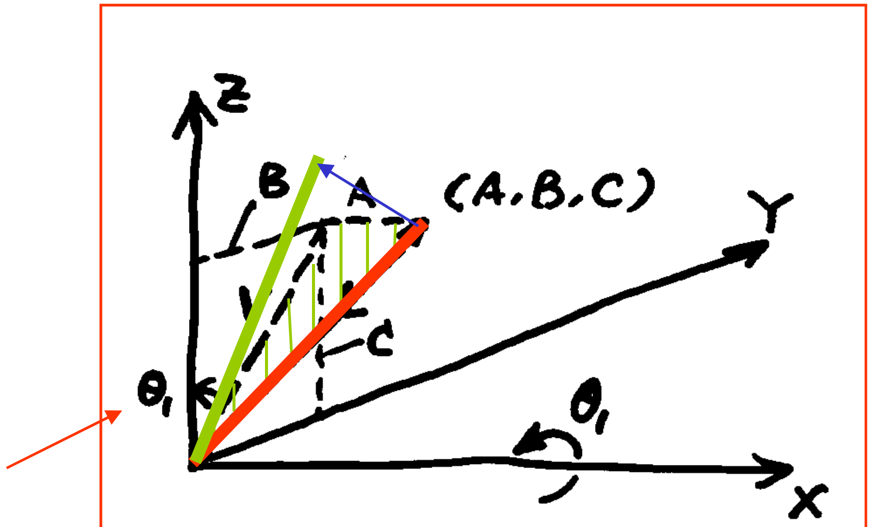
Step 1: Translate P_0 to Origin O

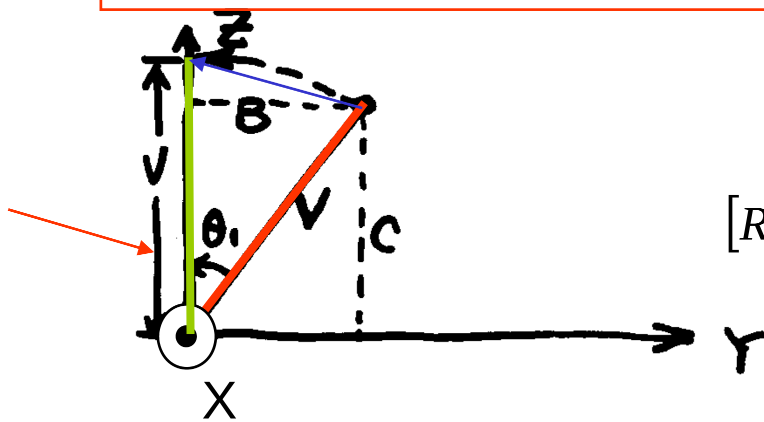
$$P_0 = [x_o \; y_o \; z_o]^T$$

$$[D] = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Step 2: Rotate Vector about X Axis to get into the x - z plane





$$L = \sqrt{A^2 + B^2 + C^2}$$

$$V = \sqrt{B^2 + C^2}$$

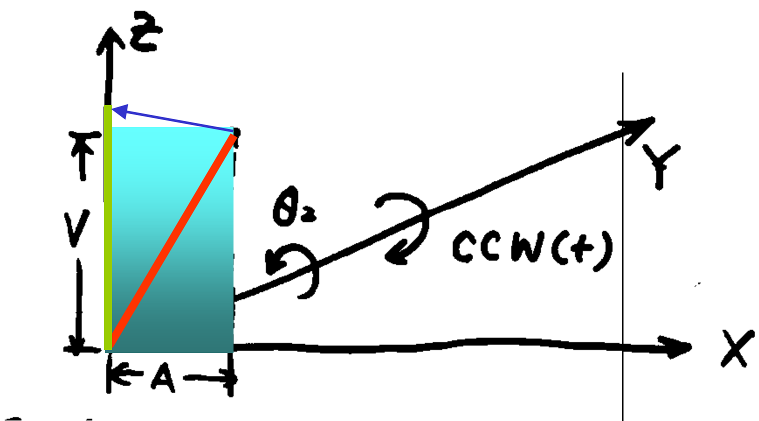
$$\sin\theta_1 = \frac{B}{V}$$

$$\cos\theta_1 = \frac{C}{V}$$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C}{V} & -\frac{B}{V} & 0 \\ 0 & \frac{B}{V} & \frac{C}{V} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Rotate about the Y axis to get it in the *Z* direction

Rotate a negative angle (CW)!

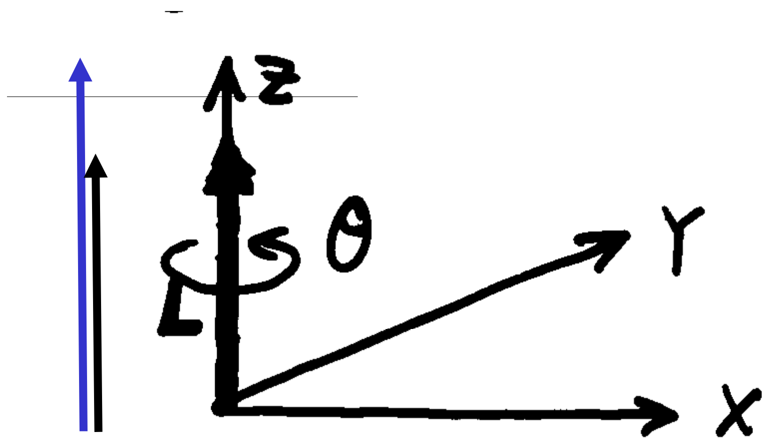


$$\sin\theta_2 = -\frac{A}{L}$$

$$\cos\theta_2 = \frac{V}{L}$$

$$[R_y] = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{V}{L} & 0 & -\frac{A}{L} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{A}{L} & 0 & \frac{V}{L} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 4: Rotate angle θ about axis $\vec{L}$



$$[R_z] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Reverse the rotation about the Y axis

$$[R_y]^{-1} = \begin{bmatrix} \frac{V}{L} & 0 & \frac{A}{L} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{A}{L} & 0 & \frac{V}{L} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse of Rotation:

Replace	θ	by	$-\theta$	
	$\sin\theta$	by	$-\sin\theta$	
	$\cos\theta$	remains	$\cos\theta$	(why?)

Step 6: Reverse rotation about the X axis

$$[R_x]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C}{V} & \frac{B}{V} & 0 \\ 0 & -\frac{B}{V} & \frac{C}{V} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C}{V} & -\frac{B}{V} & 0 \\ 0 & \frac{B}{V} & \frac{C}{V} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 7: Reverse translation

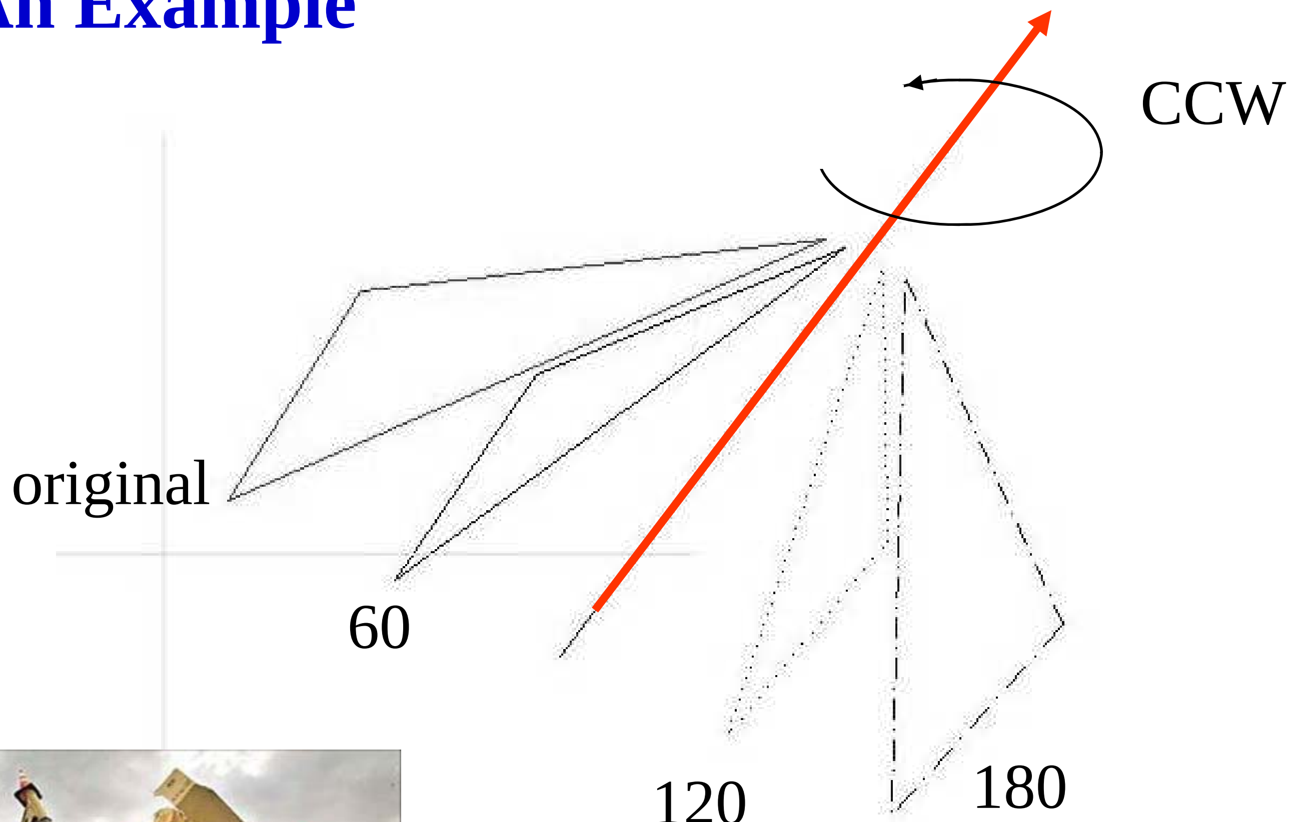
$$[D]^{-1} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Overall Transformation

$$[T] = [D]^{-1}[R_x]^{-1}[R_y]^{-1}[R_z^\theta][R_y][R_x][D]$$

$$P_2 = [T]P_1$$

An Example



An Example

Given the point matrix (four points) on the right; and a line, *NM*, with point *N* at (6, -2, 0) and point *M* at (12, 8, 0).

Rotate the these four points 60 degrees around line *NM* (alone the *N* to *M* direction) *N: u=0; M: u=1*

$$P_o = N$$
$$P_1 = M$$

$$\begin{aligned} A &= 12 - 6 = 6 \\ B &= 8 - (-2) = 10 \\ C &= 0 - 0 = 0 \end{aligned}$$

$$\begin{matrix} & \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4 \end{matrix}$$

$$\boldsymbol{P}_1 = \begin{pmatrix} 3 & 10 & 1 & 3 \\ 5 & 6 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

1. **Calculate the constants** (the Line/Axis of Rotation)

$$x = 6 + 6u$$
$$y = -2 + 10u$$
$$z = 0$$

Thus

$$A = 6,\ B = 10,\ C = 0$$

$$L = \sqrt{A^2 + B^2 + C^2} = 11.6619$$

$$V = \sqrt{B^2 + C^2} = 10$$

2. Translate *N* to the origin

$$[D] = \begin{pmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Rotate about the *X* axis

$$[R]_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C/V & -B/V & 0 \\ 0 & B/V & C/V & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Rotate about the *Y* axis

$$[R]_y = \begin{pmatrix} V/L & 0 & -A/L & 0 \\ 0 & 1 & 0 & 0 \\ A/L & 0 & V/L & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5. Rotate 60 degree (positive)

$$[R]_z = \begin{pmatrix} \cos(60) & -\sin(60) & 0 & 0 \\ \sin(60) & \cos(60) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Reverse *[R]_y*

$$[R]_y^{-1} = \begin{pmatrix} V/L & 0 & A/L & 0 \\ 0 & 1 & 0 & 0 \\ -A/L & 0 & V/L & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7. Reverse *[R]_x*

$$[R]_x^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C/V & B/V & 0 \\ 0 & -B/V & C/V & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

8. Reverse the Translation

$$[D]^{-1} = \begin{pmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

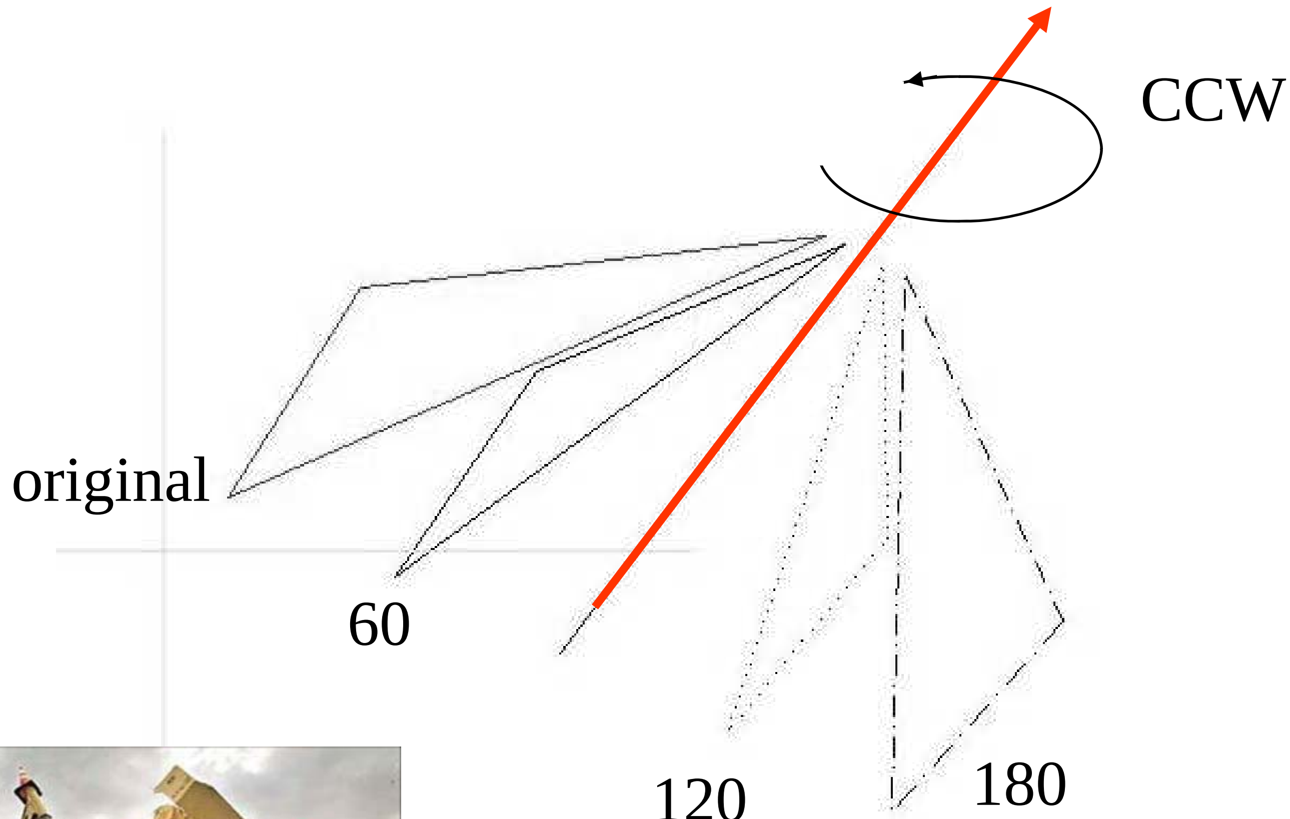
9. Calculate the total transformation

$$[T] = [D]^{-1}[R_x]^{-1}[R_y]^{-1}[R_z^{60}][R_y][R_x][D]$$

$$P_2 = [T]P_1$$

$$[P]_2 = \begin{pmatrix} 5.6471 & 10.2941 & 3.5000 & 5.6471 \\ 3.4118 & 5.8235 & -0.5000 & 3.4118 \\ 5.3468 & 0.5941 & 5.0498 & 5.3468 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 \end{pmatrix}$$

p_1 p_2 p_3 p_4


CCW
original
60
120
180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