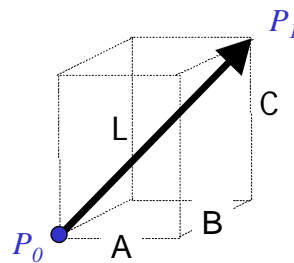
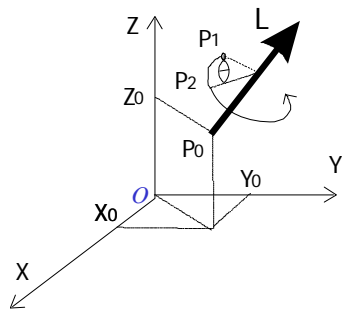


Rotation about an Arbitrary Axis (Line)



Rotation about an Arbitrary Axis (Line)

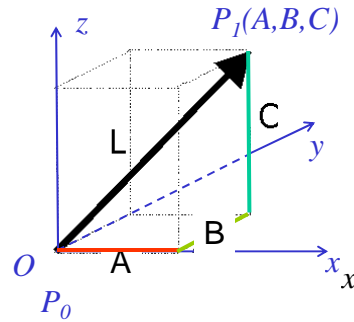


$$\begin{aligned}
 x &= Au + x_0 \\
 y &= Bu + y_0 \quad 0 \leq u \leq 1 \\
 z &= Cu + z_0 \\
 L &= \sqrt{A^2 + B^2 + C^2}u
 \end{aligned}$$

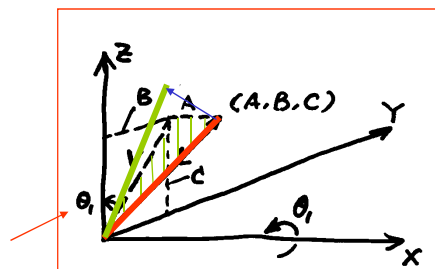
Step 1: Translate P_0 to Origin O

$$P_0 = [x_o \ y_o \ z_o]^T$$

$$[D] = \begin{bmatrix} 1 & 0 & 0 & -x_o \\ 0 & 1 & 0 & -y_o \\ 0 & 0 & 1 & -z_o \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Step 2: Rotate Vector about X Axis to get into the x - z plane

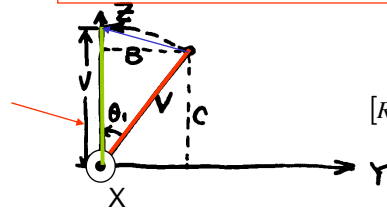


$$L = \sqrt{A^2 + B^2 + C^2}$$

$$V = \sqrt{B^2 + C^2}$$

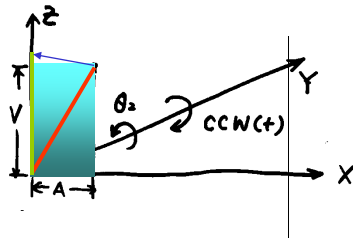
$$\sin \theta_1 = \frac{B}{V}$$

$$\cos \theta_1 = \frac{C}{V}$$



$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 & 0 \\ 0 & \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C}{V} & -\frac{B}{V} & 0 \\ 0 & \frac{B}{V} & \frac{C}{V} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Rotate about the Y axis to get it in the Z direction
 Rotate a negative angle (CW)!

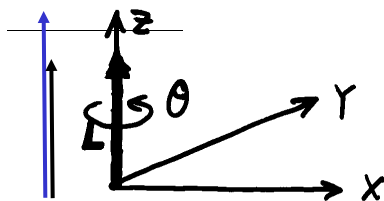


$$\sin \theta_2 = -\frac{A}{L}$$

$$\cos \theta_2 = \frac{V}{L}$$

$$[R_y] = \begin{bmatrix} \cos \theta_2 & 0 & \sin \theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{V}{L} & 0 & -\frac{A}{L} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{A}{L} & 0 & \frac{V}{L} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

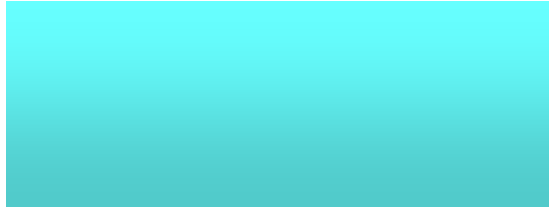
Step 4: Rotate angle θ about axis \vec{x}



$$[R_z] = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Reverse the rotation about the Y axis

$$[R_y]^{-1} = \begin{bmatrix} \frac{V}{L} & 0 & \frac{A}{L} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{A}{L} & 0 & \frac{V}{L} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Inverse of Rotation:

Replace θ by $-\theta$
 $\sin \theta$ by $-\sin \theta$
 $\cos \theta$ remains $\cos \theta$ (why?)

Step 6: Reverse rotation about the X axis

$$[R_x]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C}{V} & \frac{B}{V} & 0 \\ 0 & -\frac{B}{V} & \frac{C}{V} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 & 0 \\ 0 & \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C}{V} & -\frac{B}{V} & 0 \\ 0 & \frac{B}{V} & \frac{C}{V} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 7: Reverse translation

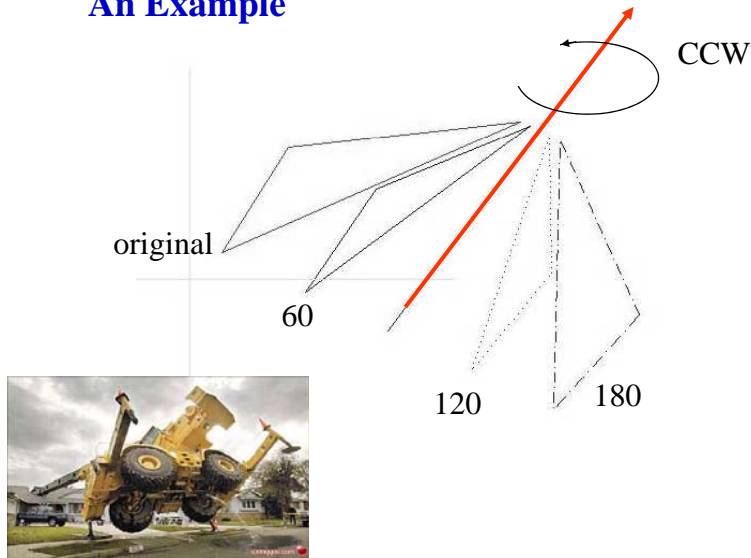
$$[D]^{-1} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Overall Transformation

$$[T] = [D]^{-1}[R_x]^{-1}[R_y]^{-1}[R_z^\theta][R_y][R_x][D]$$

$$P_2 = [T]P_1$$

An Example



An Example

Given the point matrix (four points) on the right; and a line, NM , with point N at $(6, -2, 0)$ and point M at $(12, 8, 0)$.

Rotate these four points 60 degrees around line NM (along the N to M direction) $N: u=0$; $M: u=1$

$$P_0 = N$$

$$P_1 = M$$

$A = 12 - 6 = 6$
$B = 8 - (-2) = 10$
$C = 0 - 0 = 0$

$$P_1 = \begin{matrix} P_1 & P_2 & P_3 & P_4 \\ \begin{pmatrix} 3 & 10 & 1 & 3 \\ 5 & 6 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{matrix}$$

1. Calculate the constants (the Line/Axis of Rotation)

$$x = 6 + 6u$$

$$y = -2 + 10u$$

$$z = 0$$

Thus

$$A = 6, B = 10, C = 0$$

$$L = \sqrt{A^2 + B^2 + C^2} = 11.6619$$

$$V = \sqrt{B^2 + C^2} = 10$$

2. Translate N to the origin **3. Rotate about the X axis**

$$[D] = \begin{pmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[R]_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C/V & -B/V & 0 \\ 0 & B/V & C/V & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Rotate about the Y axis **5. Rotate 60 degree (positive)**

$$[R]_y = \begin{pmatrix} V/L & 0 & -A/L & 0 \\ 0 & 1 & 0 & 0 \\ A/L & 0 & V/L & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[R]_z = \begin{pmatrix} \cos(60) & -\sin(60) & 0 & 0 \\ \sin(60) & \cos(60) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Reverse $[R]_y$

$$[R]_y^{-1} = \begin{pmatrix} V/L & 0 & A/L & 0 \\ 0 & 1 & 0 & 0 \\ -A/L & 0 & V/L & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7. Reverse $[R]_x$

$$[R]_x^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C/V & B/V & 0 \\ 0 & -B/V & C/V & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

8. Reverse the Translation

$$[D]^{-1} = \begin{pmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

9. Calculate the total transformation

$$[T] = [D]^{-1}[R_x]^{-1}[R_y]^{-1}[R_z^{60}][R_y][R_x][D]$$

$$P_2 = [T]P_1$$

$$[P]_2 = \begin{pmatrix} 5.6471 & 10.2941 & 3.5000 & 5.6471 \\ 3.4118 & 5.8235 & -0.5000 & 3.4118 \\ 5.3468 & 0.5941 & 5.0498 & 5.3468 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 \end{pmatrix}$$

P_1

P_2

P_3

P_4

