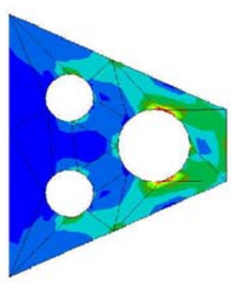
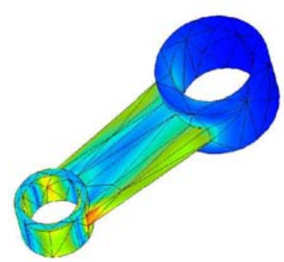
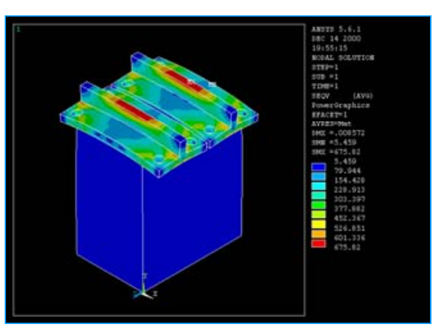
Introduction to Design Optimization

Various Design Objectives

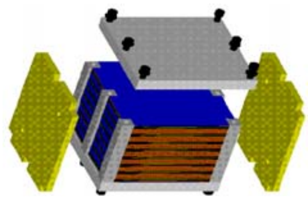
Minimum Weight (under Allowable Stress)

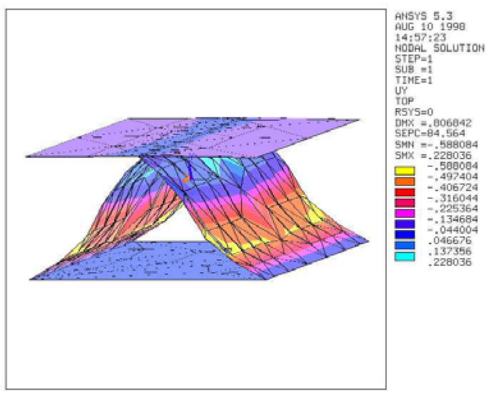






A PEM Fuel Cell Stack with Even Compression over Active Area (Minimum Stress Difference)

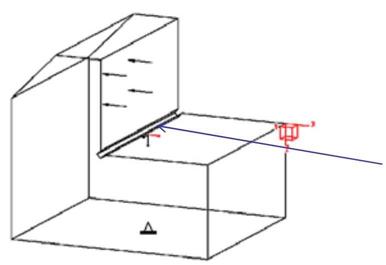




A PEM Fuel Cell Stack Multi-Functional Panel with Ideal Stiffness – to Accommodate Thermal- and Hydro-Expansions

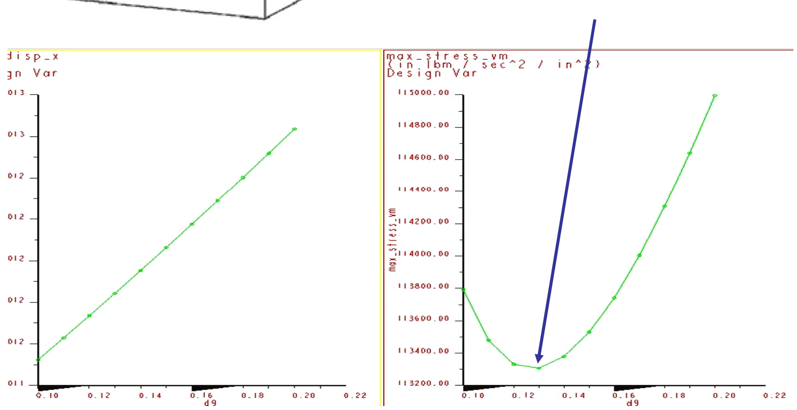
(Minimum Difference between Ideal Stiffness and Calculated Stiffness):

Find a panel design with the ideal stiffness.



Minimum Maximum Stress in the Structure

Optimized Groove Dimension to Avoid Stress Concentration or Weakening of the Structure



Engineering Applications of Optimization

- **Design** - determining design parameters that lead to the best "performance" of a mechanical structure, device, or system.

  "Core of engineering design, or the systematic approach to design" (Arora, 89)
- **Planning**
  - production planning - minimizing manufacturing costs
  - management of financial resources - obtaining maximum profits
  - task planning (robot, traffic flow) - achieving best performances
- **Control and Manufacturing** - identifying the optimal control parameters for the best performance (machining, trajectory, etc.)
- **Mathematical Modeling** - curve and surface fitting of given data with minimum error

Commonly used tool: OPT function in FEA; MATLAB Optimization Toolbox

What are common for an optimization problem?

- There are multiple solutions to the problem; and the optimal solution is to be identified.
- There exist one or more objectives to accomplish and a measure of how well these objectives are accomplished (measurable performance).
- Constraints of different forms (hard, soft) are imposed.
- There are several key influencing variables. The change of their values will influence (either improve or worsen) the "measurable performance" and the degree of violation of the "constraints."

Solution Methods

- Optimization can provide either
  - a closed-form solution, or
  - a numerical solution.
- Numerical optimization systematically and efficiently adjusts the influencing variables to find the solution that has the best performance, satisfying given constraints.
- Frequently, the design objective, or cost function cannot be expressed in the form of simple algebra. Computer programs have to be used to carryout the evaluation on the design objective or costs. For a given design variable, α, the value of the objective function, $f(\alpha)$, can only be obtained using a numerical routine. In these cases, optimization can only be carried out numerically.

α → Computer Program (no simple algebra) → $f(\alpha)$

e.g. Minimize the maximum stress in a tents/tension structures using FEA.

Definition of Design Optimization

An optimization problem is a problem in which certain parameters (*design variables*) needed to be determined to achieve the best measurable performance (*objective function*) under given *constraints*.

Classification of the Optimization Problems

- **Type of design variables**
  - optimization of continuous variables
  - integer programming (discrete variables) (examples)
  - mixed variables
- **Relations among design variables**
  - nonlinear programming $e.g.\ f(X) = Ae^{-x_1} + Bx_2$
  - linear programming $e.g.\ f(X) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$
- **Type of optimization problems**
  - unconstrained optimization
  - constrained optimization (examples)
- **Capability of the search algorithm**
  - search for a local minimum
  - global optimization; multiple objectives; etc.

Automation and Integration

- Formulation of the optimization problems
  - specifying design objective(s)
  - specifying design constraints
  - identifying design variables
- Solution of the optimization problems
  - selecting appropriate search algorithm
  - determining start point, step size, stopping criteria
  - interpreting/verifying optimization results
- Integration with mechanical design and analysis
  - **black box** analysis functions serve as objective and constraint functions (e.g. FEA, CFD models)
  - incorporating optimization results into design

An Example Optimization Problem

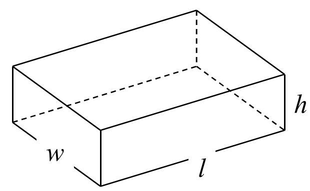
Design of a thin wall tray:

The tray has a specific volume, V, and a given height, H. The design problem is to select the length, l, and width, w, of the tray.

Given $$lwh = V \qquad h = H$$

A "workable design": $$lw = \frac{V}{H}$$

Pick either l or w and solve for others



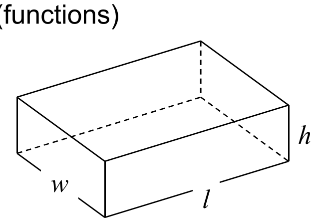
An "Optimal Design"

- The design is to minimize material volume, (or weight), where "T" is an acceptable small value for wall thickness.

Minimize $$V_m(w,l,h) = T(\underbrace{wl}_{bottom} + \underbrace{2lh + 2wh}_{sides})$$

subject to $$\left.\begin{array}{r} lwh = V \\ h = H \\ l \geq 0 \\ w \geq 0 \end{array}\right\}$$ constraints (functions)

Design variables: w, l, and h.



Standard Mathematical Form

$$\min_{w.r.t.\, l,w,h} \quad T(wl + 2lh + 2wh)$$ - objective function

Subject to

$$\left.\begin{aligned} lwh - V &= 0 \\ h - H &= 0 \end{aligned}\right\}$$ - equality constraints

- inequality constrains

$$\left.\begin{aligned} -l &\le 0 \\ -w &\le 0 \end{aligned}\right\}$$ - variable bounds

$$\vec{x} = [l, w, h]^T$$ - design vector

- for use of any available optimization routines

Analytical (Closed Form) Solution

- Eliminate the equality constrains, convert the original problem into a single variable problem, then solve it.

  from $h = H \;\&\; l\,w\,H = V;$ solve for l: $l = \dfrac{V}{Hw}$

  thus

$$\min_{w} T\left(\frac{V}{Hw}w + 2\frac{V}{Hw}H + 2wH\right) \longrightarrow \min_{w} T\left(\frac{V}{H} + 2\frac{V}{w} + 2wH\right) = f(w)$$

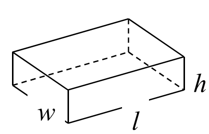
  from $\dfrac{df(w)}{dw} = 0,$ *we have* $w^2 = \dfrac{V}{H}$, *then the design optimum* $w^* = \sqrt{\dfrac{V}{H}}$

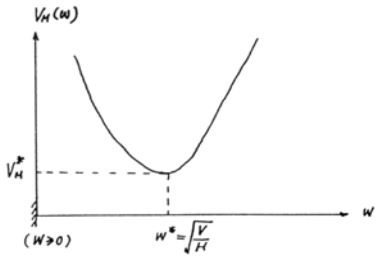
  - a stationary point

- Discard the negative value, since the inequality constraint is violated.
- The optimal value for l:

$$l^* = \frac{V}{Hw^*} = \sqrt{\frac{V}{H}} = w^*$$

$$V_M^* = T\left(\frac{V}{H} + 2hw^* + \frac{2V}{w^*}\right) = T\left(\frac{V}{H} + 4\sqrt{VH}\right)$$



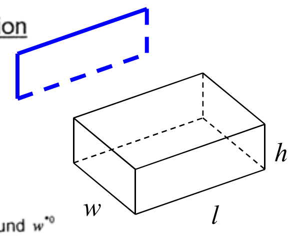


$w \leq \infty$ or $l \geq 0$

Graphical Solution

no width & length limitations
no violated constraints.

<u>Change of Constraints and Their Influence to the Final Solution</u>



Consider a modified problem:

$$\min_{l,w,h} \ V_m = T\,(w \times l + 2 \times l \times h + 2 \times w \times h)$$

s.t.

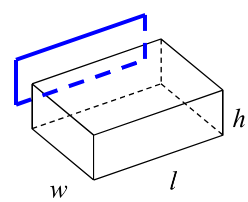
$$lwh = V$$
$$h = H$$
$$w \geq 0$$
$$l \geq 0$$

Handled as an unconstrained problem and found w^{*0}

$w \leq W$ maximum width / add a new constraint

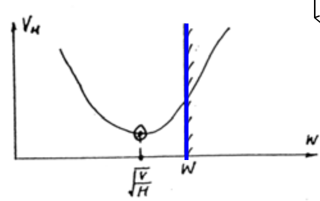
Follow the previous example:

unconstrained optimum: $w^{*0} = \sqrt{\frac{V}{H}}$



- For $W \geq w^{*0} = \sqrt{\frac{V}{H}}$

  $w^* = l^* = \sqrt{\frac{V}{H}}$

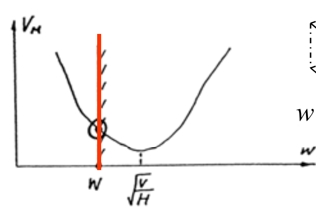


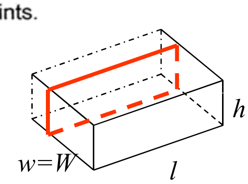
The constrained optimum is not changed, no active constraints.

- For $W \leq w^{*0} = \sqrt{\frac{V}{H}}$

  $w^* = W \neq w^{*0}$

  $l^* = \frac{V}{HW}$





Constraint $w \leq W$ is "active."

Procedures for Solving an Eng. Optimization Problem

- **Formulation of the Optimization Problem**
  - Simplifying the physical problem
  - identifying the major factor(s) that determine the performance or outcome of the physical system, such as costs, weight, power output, etc. – objective
  - Finding the primary parameters that determine the above major factors - the design variables
  - Modeling the relations between design variables and the identified major factor - objective function
  - Identifying any constraints imposed on the design variables and modeling their relationship – constraint functions
- **Selecting the most suitable optimization technique or algorithm to solve the formulated optimization problem.**

  - requiring an in-depth know-how of various optimization techniques.
- **Determining search control parameters**

  - determining the initial points, step size, and stopping criteria of the numerical optimization
- **Analyzing, interpreting, and validating the calculated results**

  An optimization program does not guarantee a correct answer, one needs to
  - prove the result mathematically.
  - verify the result using check points.

Standard Form for Using Software Tools for Optimization (e.g. MatLab Optimization Tool Box)

Minimize $f(\vec{x})$ (1)

with respect to $\vec{x} = (x_1\ x_2 \cdots\cdots\cdots x_n)$

subject to

$h_k(\vec{x}) = \{0\}$ $\quad k = 1, \cdots, p$ (2)

$g_j(\vec{x}) \le \{0\}$ $\quad j = 1, \cdots, q$ (3)

$x_i^{(l)} \le x_i \le x_i^{(u)}$ $\quad i = 1, \cdots, N$ (4)

$f(\vec{x})$ —— a real valued function (objective function)

$\vec{x}$ —— a N-component vector (design variables)

$\varepsilon_g 2$ —— equality constraints

$\varepsilon_g 3$ —— inequality constraints

$\varepsilon_g 4$ —— variable founds

Use of MATLAB Optimization Toolbox

Notes

- A maximization problem can be converted into a minimization problem by:

$$\max \; f(\vec{x}) \Rightarrow \min \frac{1}{f(\vec{x})} \quad \text{or} \quad \min\left\{-f(\vec{x})\right\}$$

- $\varepsilon_g 4$ can be converted into $\varepsilon_g 3$:

$$x_i - x_i^{(u)} \le 0$$
$$x_i^{(l)} - x_i \le 0$$

adding *2N* inequality constraints

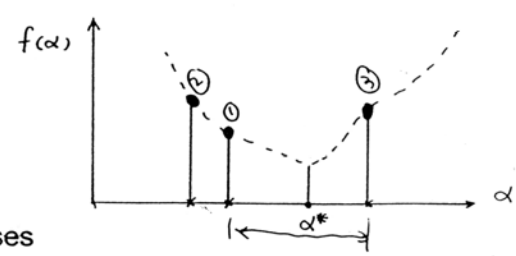
Assignment: (notebook only, not to turn in)

- Consider a circular tray, find the minimum v_m with $h = H$, and also with h free. The diameter of the tray, d, is a design variable.
- Compare the above two "competing" design in terms of v_m.

One Dimensional Search Methods

The 1-D Search Problem - Basis for ND Optimization Search Techniques

$\min\limits_{w.r.t.\,\alpha} f(\alpha)$



In many cases

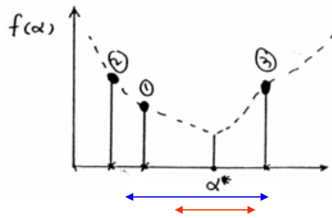
Often Black-box Function

α → **Computer Program (no simple algebra)** → *f(α)*

an example: **tent design**

(maximum stress less than allowed value)

1-D Optimization



- The Search
  - We don't know the curve. Given α, we can calculate $f(\alpha)$.
  - By inspecting some points, we try to find the approximated shape of the curve, and to find the minimum, α^* numerically, using as few function evaluation as possible.
- The procedure can be divided into two parts:

  a) Finding the "range" or region " known " to contain α^* .

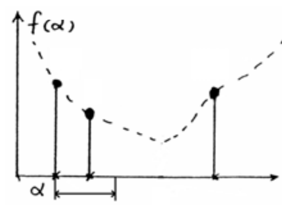
  b) Calculating the value of α^* as accurately as designed or as possible within the range – narrowing down the range.

Search Methods

- Typical approaches include:
  - Quadratic Interpolation (Interpolation Based)
  - Cubic Interpolation
  - Newton-Raphson Scheme (Derivative Based)
  - Fibonacei Search (Pattern Search Based)
  - Golden Section Search
- Iterative Optimization Process:
  - Start point α_o → OPTIMIZATION → Estimated point α_k → New start point α_{k+1}
  - Repeat this process until the stopping rules are satisfied, then $\alpha^* = \alpha_n$.

Iterative Process for Locating the Range

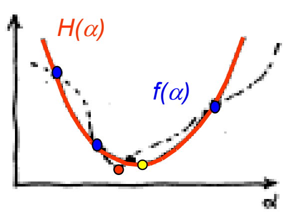
- Picking up a start point, $\alpha_{o,}$ and a range;
- Shrinking the range;
- Doubling the range;
- Periodically changing the sign.



Typical Stooping Rules:

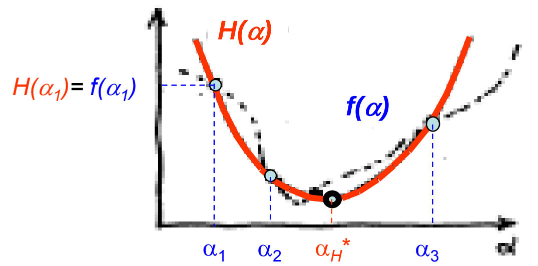
$$\frac{\left|f(new\ \alpha^*) - f(last\ \alpha^*)\right|}{\left|f(new\ \alpha^*)\right|} < \varepsilon \quad or \quad \frac{\left|new\ \alpha^* - last\ \alpha^*\right|}{new\ \alpha^*} < \varepsilon$$

Quadratic Interpolation Method



$$f(\alpha) \Leftarrow H(\alpha) = a + b\alpha + c\alpha^2$$

- Quadratic Interpolation uses a quadratic function, $H(\alpha)$, to approximate the "unknown" objective function, $f(\alpha)$.
- Parameters of the quadratic function are determined by several points of the objective function, $f(\alpha)$.
- The known optimum of the interpolation quadratic function is used to provide an estimated optimum of the objective function through an iterative process.
- The estimated optimum approaches the true optimum.
- The method requires proper range being found before started.
- It is relatively efficient, but sensitive to the shape of the objective



assume $f(\alpha) \approx H(\alpha) = a + b\alpha + c\alpha^2$

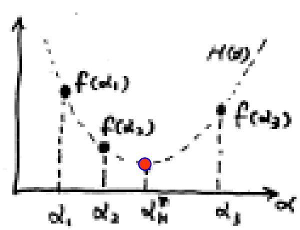
$$\frac{dH(\alpha)}{d\alpha} = b + 2c\alpha = 0$$

$$\alpha_H^* = -\frac{b}{2c}$$

α_H^* is the optimum for $H(\alpha)$, and is used to estimate the optimum for $f(\alpha)$.

To find parameters a, b, & c, three function evaluations are required at $\alpha_1, \alpha_2, \alpha_3$:

$$a + b\alpha_1 + c\alpha_1^2 = f(\alpha_1) = f_1$$
$$a + b\alpha_2 + c\alpha_2^2 = f(\alpha_2) = f_2$$
$$a + b\alpha_3 + c\alpha_3^2 = f(\alpha_3) = f_3$$



$$b = \frac{(f_1 - f_2) - c(\alpha_1^2 - \alpha_2^2)}{\alpha_1 - \alpha_2}$$

$$c = \frac{(f_1 - f_3)(\alpha_1 - \alpha_2) - (f_1 - f_2)(\alpha_1 - \alpha_3)}{(\alpha_1^2 - \alpha_3^2)(\alpha_1 - \alpha_2) - (\alpha_1^2 - \alpha_2^2)(\alpha_1 - \alpha_3)}$$

numerator – 2

denominator - 1

$$\alpha_H^* = -\frac{b}{2c}$$ — necessary condition for the optimum of $H(\alpha)$.

Now, chose α_1, α_2, α_3 to satisfy the sufficient condition for α_H^* to be the optimum of $H(\alpha)$

$$\frac{d^2H(\alpha)}{d\alpha^2} = 2C > 0 \quad \text{or} \quad C > 0$$

Consider $0 \le \alpha_1 < \alpha_2 < \alpha_3$

— α value chosing law

· The denominator of C:

$$(\alpha_1-\alpha_3)(\alpha_1-\alpha_2)[(\alpha_1+\alpha_3)-(\alpha_1+\alpha_2)]$$

$$= (\alpha_3-\alpha_1)(\alpha_2-\alpha_1)(\alpha_3-\alpha_2) > 0$$

· The numerator of C:

$$N = (\alpha_3-\alpha_1)(f_1-f_2) - (\alpha_2-\alpha_1)(f_1-f_3)$$

$$\because (\alpha_3-\alpha_1) > (\alpha_2-\alpha_1)$$

if $(f_1-f_2) > (f_1-f_3)$ and $(f_1-f_2) > 0$, $N > 0$

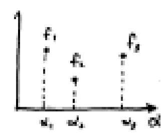
or $C > 0$

Condition: $f_1 > f_2$ and $f_2 < f_3$

Range required to find the optimum using quadratic interpolation:

$0 \le \alpha_1 < \alpha_2 < \alpha_3$

$f_1 > f_2 < f_3$



Algrithm:

Point update schemes based on the relations between the center point, α_2, $f(\alpha_2)$, and the present optimum: α^*, $f(\alpha^*)$.

After setting up the 3 points from Range Finding, carry out an interpolation to calculate α_H^* & $f(\alpha_H^*)$

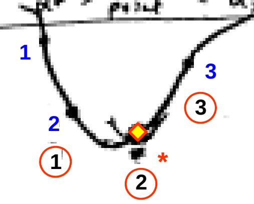
- relative mag. of $f(\alpha^*)$ > center < $f(\alpha_2)$
- relative pos. of α^* > point < α_2

1) If $f(\alpha_H^*) < f(\alpha_2)$

and $\alpha_2 < \alpha_H^* < \alpha_3$

Let $\alpha_1 = \alpha_2$, $\alpha_2 = \alpha_H^*$

$\alpha_3 = \alpha_3$ and recompute α_H^*

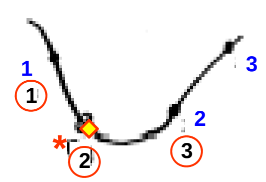


2) If $f(\alpha_H^*) < f(\alpha_2)$

and $\alpha_1 < \alpha_H^* < \alpha_2$

Let $\alpha_1 = \alpha_1$, $\alpha_2 = \alpha_H^*$

$\alpha_3 = \alpha_2$ and recompute α_H^*

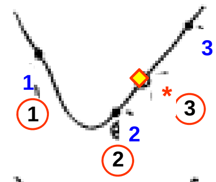


3) If $f(\alpha_H^*) > f(\alpha_2)$

and $\alpha_2 < \alpha_H^* < \alpha_3$

Let $\alpha_1 = \alpha_1$, $\alpha_2 = \alpha_2$

$\alpha_3 = \alpha_H^*$ and recompute α_H^*

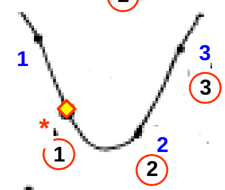


4) If $f(\alpha_H^*) > f(\alpha_2)$

and $\alpha_1 < \alpha_H^* < \alpha_2$

Let $\alpha_1 < \alpha_H^*$, $\alpha_2 = \alpha_2$

$\alpha_3 = \alpha_3$ and recompute α_H^*



Repeat, until a "stopping rule" is satisfied.

NOTE: In 3) & 4), Restart is required for efficient convergence, if this happens repeatedly.

Restart: a range operation (new) about α using,

$t_0 = \min[\,|\alpha_3 - \alpha_2|, |\alpha_2 - \alpha_1|\,]$

III. N - D Search

N Dimensional Optimization Methods

- Non Gradient Based Search Schemes
  - Univariate Search
  - Direct Search (Hunt & Pack)
  - Random Search
  - Conjugate Direction Search (Powell)] Pattern Search
  - Method of Hook & Jeeves
- Gradient Based Search Schemes
  - Steepest Descent
  - Conjugate Gradient Search (Fletcher & Reeves)
  - Newton's Method
  - Quasi - Newton Methods
    - DFPM (Davidon, Fletcher, Powell Method)
    - BFGS (Broyden, Fletcher, Goldfarb, Shanno)

The non-gradient based search schemes work better with ill-behaved objective functions. They are less efficient. The gradient based search schemes are more efficient, but they are more sensitive to the shape of the objective function.

N-Dimensional Search

- The Problem now has N Design Variables.
- Solving the Multiple Design Variable Optimization (Minimization) Problem Using the 1-D Search Methods Discussed Previously
- This is carried out by:
  - To choice a direction of search
    - o To deal one variable each time, in sequential order - easy, but take a long time (*e.g.* $x_1, x_2, \ldots, x_N$)
    - o To introduce a new variable/direction that changes all variables simultaneously, more complex, but quicker (*e.g.* $\boldsymbol{S}$)
  - Then to decide how far to go in the search direction (small step $\varepsilon = \Delta x$, or large step determining α by 1D search)

III N-D Search Methods

1. Non-Gradient Based Search Methods

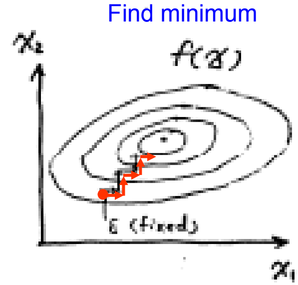
Univariate Search

Search is carried out in a sequence of fixed and prespecified directions (usually the coordinate directions).

a) Small Step Search

- Hold all x_j constant, except x_i
- $f(x_i + \epsilon) \lessgtr f(x_i)$ (new f vs. old f)
  - T → accept it
  - F → next variable
- Repeat until stopping rule satisfied.

Find minimum



Large Step 1-D Search

- Two Key Questions:
  - Search Direction – In what direction should we move (or search)?
  - Next Point – How far we should go or where do we stop along this move/search direction?
- Search along the Coordinate Direction (Univariate Search)
- Introduce a New Variable α which represents how far we should go along the selected search direction
- The value of α is determined by a 1-D optimization problem

$$\underset{w.r.t\ \alpha}{Min}\ f(\alpha)$$

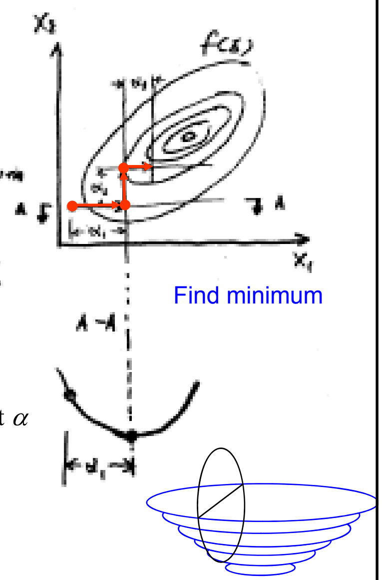
$\underline{x}_k = old\ \underline{x}$ is a known point

b) Large Step Search

- Hold all x_j constant, except x_i
- 1-D minimization in x_i direction

  New $\underline{X}$ = Old $\underline{X}$ + $\alpha \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ← ith position

- Repeat with next variable, until stopping rule satisfied.



new point: $\underline{x}_{k+1} = \underline{x}_k + \begin{bmatrix} 0 \\ 0 \\ \alpha \\ 0 \end{bmatrix}$ is almost defined except α

now in the N-D space, there is only one variable: α

Gradient Based N-D Search

- The Problem has N Design Variables.
- Solving the Multiple Design Variable Optimization (Minimization) Problem Using the 1-D Search Methods
- This is carried out by:
  - To identify the search/move direction, ***S*** – the direction that has the maximum down hill slope (the direction opposite to the directive direction)
  - To introduce a new variable α that represents how far do we move along the identified search/move direction
  - To determine this variable α by 1-D minimization using α as the design variable.

2. Gradient Based Methods

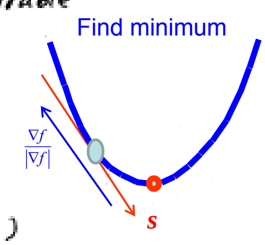
Steepest Descent

$$\underline{x}_{k+1} = \underline{x}_k + \underline{S}$$

Consider a 2-D problem where a step $\underline{S} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$. what $\underline{S}$ will make the biggest reduction in $f(\underline{x})$?

— steps in the gradient direction seem desirable

$$\underline{S} = -\alpha \frac{\nabla f}{|\nabla f|}$$



Algorithm:

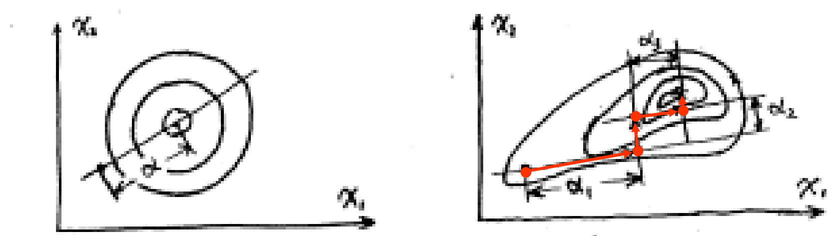
1) Evaluate ∇f at x_k.
2) Let $x_{k+1} = x_k - \alpha \frac{\nabla f}{|\nabla f|}$
3) Search α which minimizes $f(x_{k+1})$
4) After the 1-D minimum is found, reevaluate ∇f, perform a new 1-D search.
5) Stop when stopping rules are satisfied

N-D Search Problem → 1D Search Problem on Variable α

$\underline{x}_k = old\ \underline{x}$ is a known point

new point: $\underline{x}_{k+1} = \underline{x}_k - \alpha \frac{\nabla f}{|\nabla f|}$ is also known except α

now in the N-D space, there is only one variable: α



Symetric Quadratic Fcn. General Obj. Function

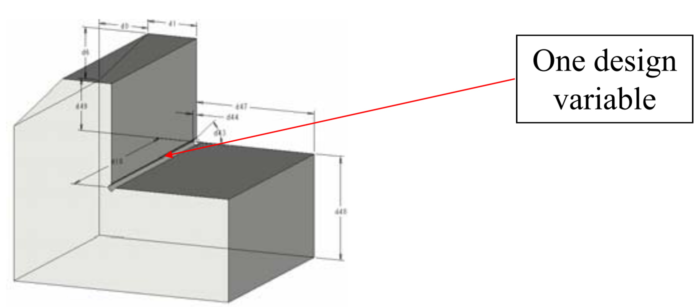
- Two consecutive steps are orthogonal to each other.

Can we search along the diagonal dir?

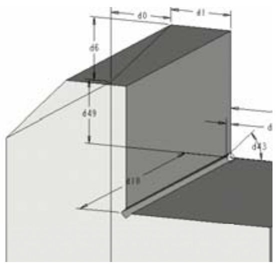
Answer to this question leads to more advanced search algorithms.

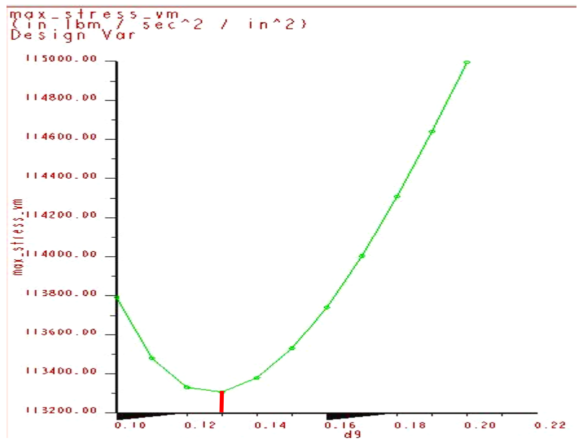
Design Optimization

Objective: minimize the maximum stress in the structure
Constraints: maximum deformation of the L bracket



Result of the Optimization



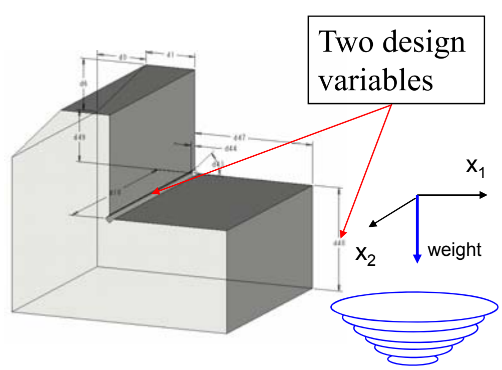


Best groove size: 0.13 (with minimum Maximum Stress)

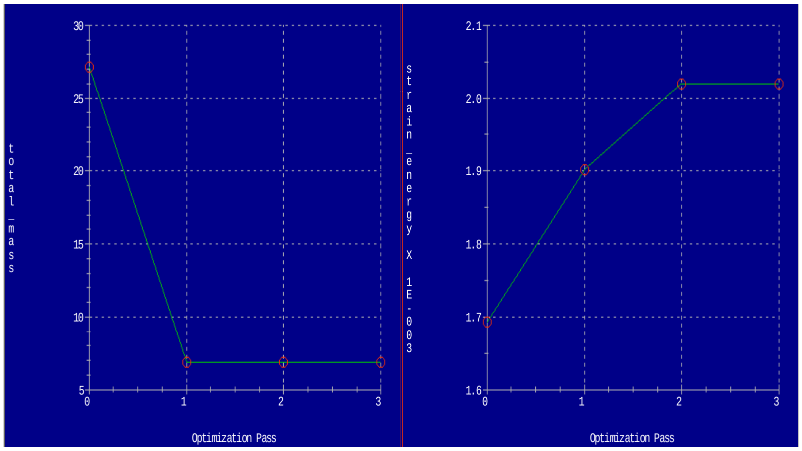
An Different Design Optimization

Objective: minimize the weight (mass) of the structure
Constraints: maximum load and deformation

1. Define relations to control the model generation (two design parameters; one is the groove size and the other is the overall fixture size.)
2. Specify ranges of variables, objective, and constraints
3. Perform the optimization (about 15 min.)
4. Results plotting and convergence check



Optimal Design



The Total Mass and Strain Energy Convergence Plots in the Optimization

Formulation of Different Design Optimization Problem

Best Performance Design – Lightest Coffee Mug

Minimize Mass of the mug as a function of mug dimensions (D: Diameter, H: Height, T: Thickness) -- ***Objective Function***

Subject to

Mug Volume ≥ A Constant	-- ***Inequality Constraint***
H/D = 1.65	-- ***Equality Constraint***
D, H, T > 0	-- ***Variables***

Find: D^*, H^*, and T^* -- ***Optimum***

Formulation of Different Design Optimization Problem

Lowest Cost Design – Cheapest Coffee Mug

Minimize Mfg. Cost of the Mug -- ***Objective Function***

Subject to

Mug Volume $\geq$ Constant 1 -- ***Inequality Constraint***

Mug Mass $\leq$ Constant 2

Strength $\geq$ Constant 3

H/D = 1.65 -- ***Equality Constraint***

D, H, T, Material, Tolerances, etc. -- ***Variables***

Find: D^*, H^*, and T^* etc. -- ***Optimum***

Are you ready for the first quiz?