Problem

Derive a parametric representation of an ellipse using the recursive approach. The parametric expression of an ellipse has the form:

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$$

Can you apply the same method to generate the involute curve of a spur gear if the curve function is given as:

$$\begin{cases} x = R\cos\theta + \theta\sin\theta \\ y = R\sin\theta + \theta\cos\theta \end{cases}$$

and explain why.

1)
- $x_n = a\cos\theta$
- $y_n = b\sin\theta$

1) • $x_n = a\cos\theta$

• $y_n = b\sin\theta$

• $x_{n+1} = a\cos(\theta + d\theta)$

$= a[\cos\theta \cdot \cos d\theta - \sin\theta \cdot \sin d\theta]$

$= x_n \cos d\theta - y_n \frac{a}{b} \sin d\theta$

• $y_{n+1} = a\sin(\theta + d\theta)$

$= a[\sin\theta \cos d\theta + \cos\theta \sin d\theta]$

$= y_n \cos d\theta + x_n \frac{b}{a} \sin d\theta$.

2) The same method cannot be used, due to the

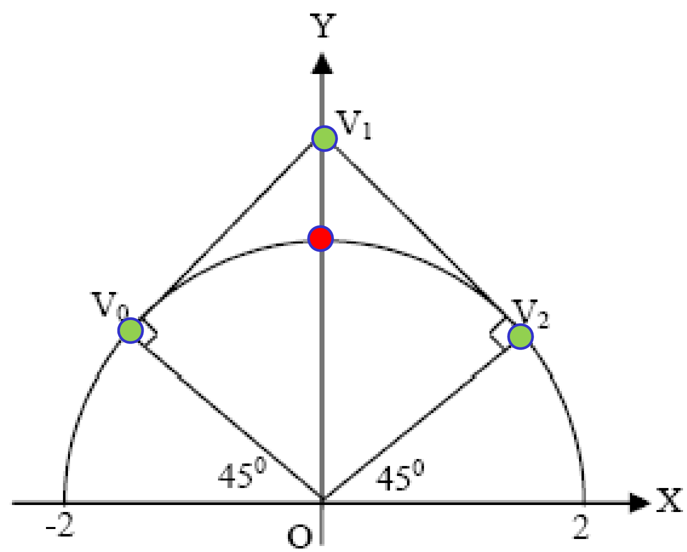
$\theta \cdot \sin\theta$ & $\theta\cos\theta$ terms.

**PROBLEM**

A Bezier curve with quadratic base functions can be specified by three points, $V_0$, $V_1$, and $V_2$. The general form of the curve is:

$$\vec{p}(u) = \sum_{i=0}^{2} V_i \cdot B_{i,2}(u) \qquad (1)$$

- Derive the Bezier curve expression of the arc shown in the figure on the right.
- Calculate $\vec{p}(0.5)$ and $\vec{p}\,'(0.5)$



- Derive the Bezier curve expression of the arc shown in the figure on the right.

From the figure, the three control points can be calculated as

$$V_0 = \begin{bmatrix} -2 \cdot \cos 45^\circ \\ 2 \cdot \sin 45^\circ \end{bmatrix} = \begin{bmatrix} -1.414 \\ 1.414 \end{bmatrix} \quad (2)$$

$$V_1 = \begin{bmatrix} 0 \\ \sqrt{2^2 + 2^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2.828 \end{bmatrix} \quad (3)$$

$$V_2 = \begin{bmatrix} 2 \cdot \cos 45^\circ \\ 2 \cdot \sin 45^\circ \end{bmatrix} = \begin{bmatrix} 1.414 \\ 1.414 \end{bmatrix} \quad (4)$$

The Bezier curve can be simplified as

$$\vec{p}(u) = V_0 \cdot B_{0,2}(u) + V_1 \cdot B_{1,2}(u) + V_2 \cdot B_{2,2}(u) \quad (5)$$

As the Bezier base functions are defined as

$$B_{i,n}(u) = \begin{pmatrix} n \\ i \end{pmatrix} \cdot u^i \cdot (1-u)^{n-i} \quad (6)$$

then

$$B_{0,2}(u) = \frac{2!}{0!(2-0)!} \cdot u^0 (1-u)^{2-0} = (1-u)^2 \quad (7)$$

$$B_{1,2}(u) = \frac{2!}{1!(2-1)!} \cdot u^1 (1-u)^{2-1} = 2 \cdot u \cdot (1-u) \quad (8)$$

$$B_{2,2}(u) = \frac{2!}{2!(2-2)!} \cdot u^2 (1-u)^{2-2} = u^2 \quad (9)$$

The Bezier curve is

$$\vec{p}(u) = (1-u)^2 \cdot V_0 + 2\cdot u\cdot(1-u)\cdot V_1 + u^2\cdot V_2 = (V_0 - 2\cdot V_1 + V_2)\cdot u^2 + 2\cdot(V_1 - V_0)\cdot u + V_0$$

$$= \begin{bmatrix} 0 \\ -2.828 \end{bmatrix}\cdot u^2 + \begin{bmatrix} 2.828 \\ 2.828 \end{bmatrix}\cdot u + \begin{bmatrix} -1.414 \\ 1.414 \end{bmatrix} \tag{10}$$

The first order derivative of the curve is

$$\vec{p}\,'(u) = 2\cdot(V_0 - 2\cdot V_1 + V_2)\cdot u + 2\cdot(V_1 - V_0)$$

$$= \begin{bmatrix} 0 \\ -5.656 \end{bmatrix}\cdot u + \begin{bmatrix} 2.828 \\ 2.828 \end{bmatrix} \tag{11}$$

- Calculate $\vec{p}(0.5)$ and $\vec{p}\,'(0.5)$

Substitute $u = 0.5$ into Eq. (10) and (11),

$$\vec{p}(0.5) = \begin{bmatrix} 0 \\ 2.828 \end{bmatrix}\cdot(0.5)^2 + \begin{bmatrix} 2.828 \\ 2.828 \end{bmatrix}\cdot(0.5) + \begin{bmatrix} -1.414 \\ 1.414 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.121 \end{bmatrix} \tag{12}$$

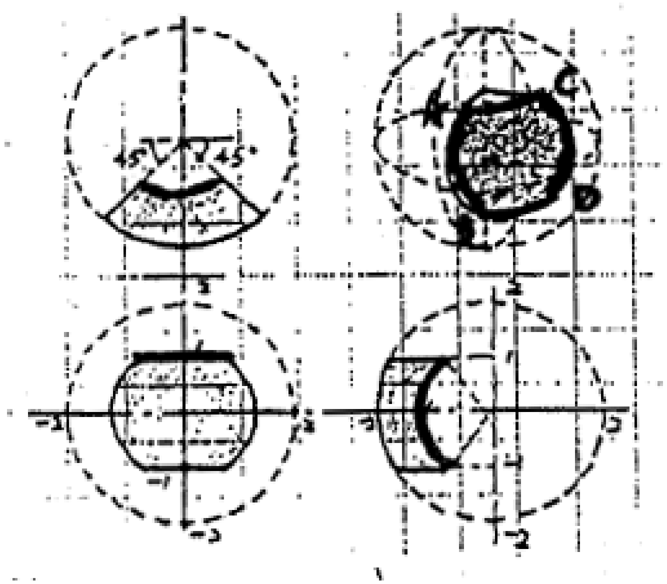
$$\vec{p}\,'(0.5) = \begin{bmatrix} 0 \\ -5.656 \end{bmatrix}\cdot(0.5) + \begin{bmatrix} 2.828 \\ 2.828 \end{bmatrix} = \begin{bmatrix} 2.828 \\ 0 \end{bmatrix} \tag{13}$$

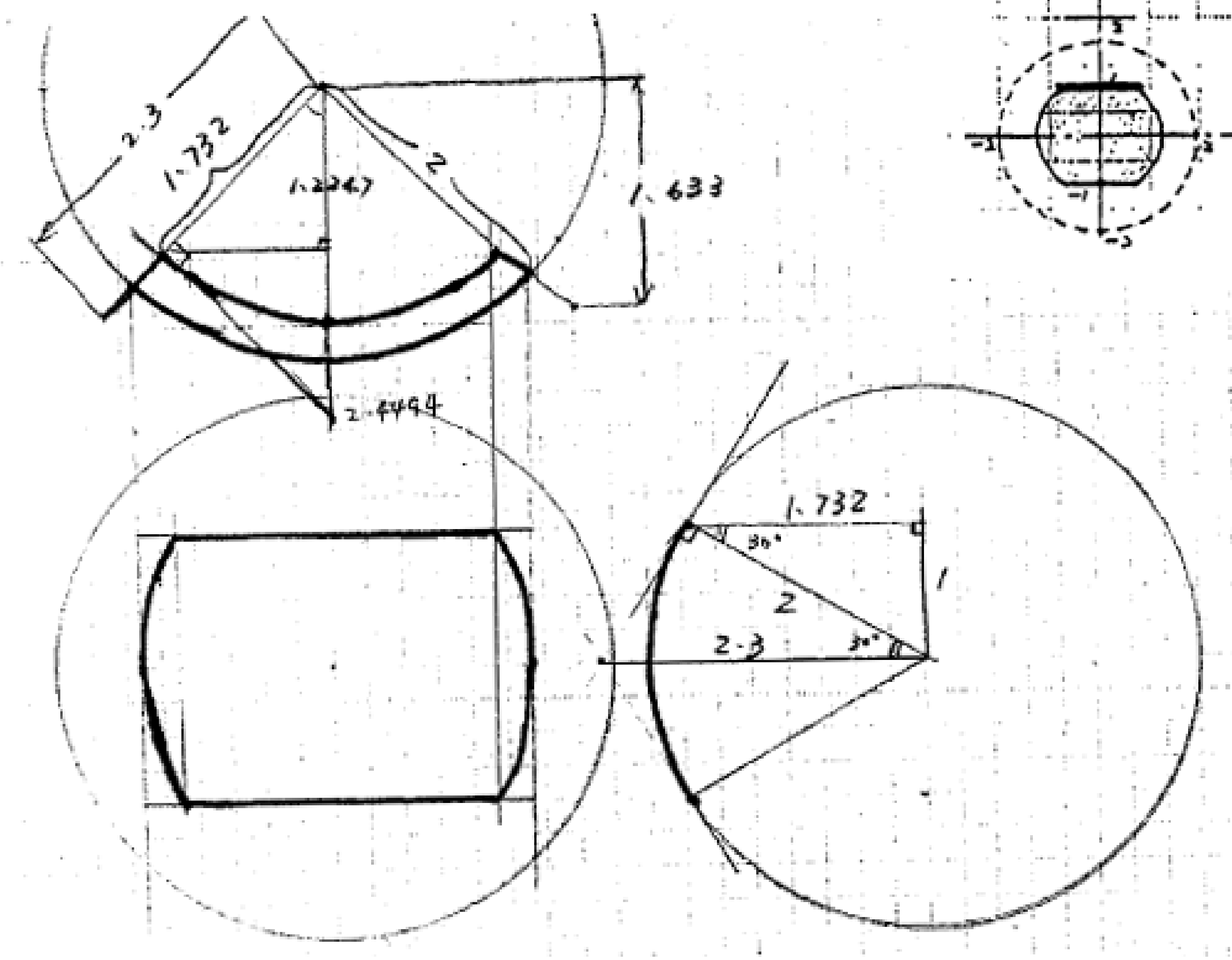
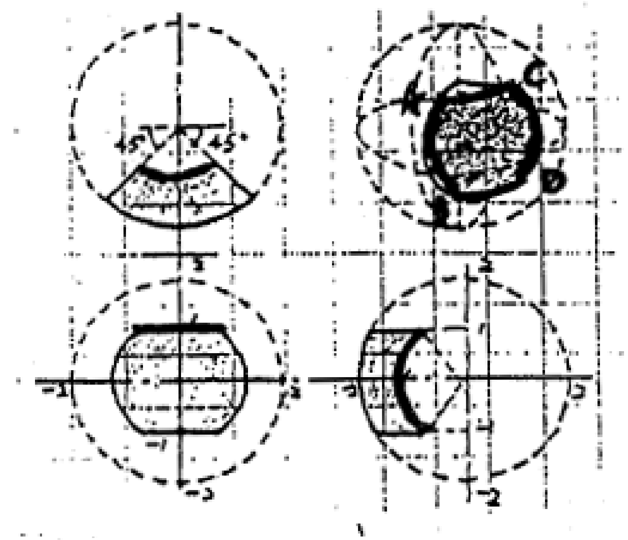
Problem

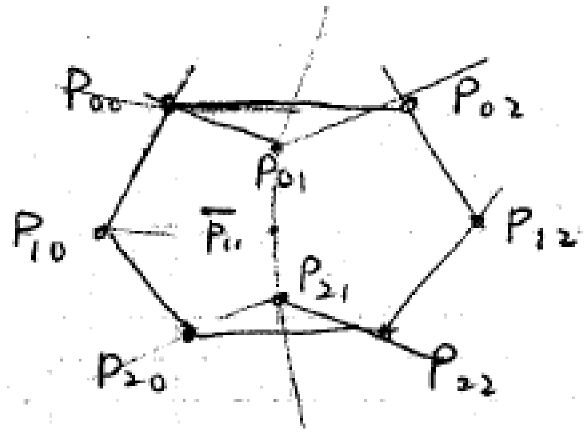
A Bezier surface patch with a quadratic basis can be specified by nine points, and has the general form:

$$p(u,w) = \begin{bmatrix} (1-u)^2 & 2u(1-u) & u^2 \end{bmatrix} \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} (1-w)^2 \\ 2w(1-w) \\ w^2 \end{bmatrix}$$

- Derive the Bezier surface expression for the gray area of sphere surface shown in the figure
- Calculate $p(0.5, 0)$, $p'_u(0.5, 0)$, and $p'_w(0.5, 0)$.
- Determine the surface normal $n(0.5, 0)$.
- If the depth of cut of the offset cutting is 0.05, determine the offset point $p(0.5, 0)_{offset}$




2.3
1.732
1.2247
2
1.633
2.4494
1.732
30°
2
1
2.3
30°

45°
45°




$$\bar{P}_{00} = \begin{bmatrix} 1.2247 \\ -1.2247 \\ 1 \end{bmatrix}$$

$$\bar{P}_{01} = \begin{bmatrix} 1.732 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{P}_{02} = \begin{bmatrix} 1.2247 \\ 1.2247 \\ 1 \end{bmatrix}$$

$$\bar{P}_{10} = \begin{bmatrix} 1.633 \\ -1.633 \\ 0 \end{bmatrix} \quad \bar{P}_{12} = \begin{bmatrix} 1.633 \\ 1.633 \\ 0 \end{bmatrix} \quad \bar{P}_{11} = \begin{bmatrix} 2.3 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{P}_{20} = \begin{bmatrix} 1.2247 \\ -1.2247 \\ -1 \end{bmatrix} \quad \bar{P}_{21} = \begin{bmatrix} 1.732 \\ 0 \\ -1 \end{bmatrix} \quad \bar{P}_{22} = \begin{bmatrix} 1.2247 \\ 1.2247 \\ 1 \end{bmatrix}$$

$$P(u, w) = \left[ (1-u)^2 \;\; 2u(1-u) \;\; u^2 \right] \begin{bmatrix} \bar{P}_{00} & \bar{P}_{01} & \bar{P}_{02} \\ \bar{P}_{10} & \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{20} & \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} \begin{bmatrix} (1-w)^2 \\ 2w(1-w) \\ w^2 \end{bmatrix}$$

$$P(u,w) = \left[(1-u)^2 \quad 2u(1-u) \quad u^2\right] \begin{bmatrix} \bar{P}_{00} & \bar{P}_{01} & \bar{P}_{02} \\ \bar{P}_{10} & \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{20} & \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} \begin{bmatrix} (1-w)^2 \\ 2w(1-w) \\ w^2 \end{bmatrix}$$

$$P'_u(u\,w) = \left[-2(1-u) \quad 2(1-2u) \quad 2u\right] \{\mathbb{P}\} \begin{bmatrix} (1-w)^2 \\ 2w(1-w) \\ w^2 \end{bmatrix}$$

$$P'_w(u,w) = \left[(1-u)^2 \quad 2u(1-u) \quad u^2\right] \{\mathbb{P}\} \begin{bmatrix} -2(1-w) \\ 2(1-2w) \\ 2w \end{bmatrix}$$

$$P(0.5,0) = \cdots$$

$$\bar{n}(0.5,0) = \frac{\bar{P}'_u(0.5,0) \times \bar{P}'_w(0.5,0)}{\left|P'(0.5,0) \times \bar{P}'_w(0.5,0)\right|}.$$

$$\bar{P}_{offset}(0.5,0) = \bar{P}(0.5,0) + 0.05 \times \bar{n}(0.5,0) = \cdots$$

# Example Problem for Finding the Bezier Curve

**Example 5.19.** The coordinates of four control points relative to a current WCS are given by

$$\mathbf{P}_0 = [2 \quad 2 \quad 0]^T, \quad \mathbf{P}_1 = [2 \quad 3 \quad 0]^T, \quad \mathbf{P}_2 = [3 \quad 3 \quad 0]^T, \quad \text{and} \quad \mathbf{P}_3 = [3 \quad 2 \quad 0]^T$$

Find the equation of the resulting Bezier curve. Also find points on the curve for $u = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1.

***Solution.*** Equation (5.91) gives

$$\mathbf{P}(u) = \mathbf{P}_0 B_{0,3} + \mathbf{P}_1 B_{1,3} + \mathbf{P}_2 B_{2,3} + \mathbf{P}_3 B_{3,3}, \qquad 0 \leq u \leq 1$$

Using Eqs. (5.92) and (5.93), the above equation becomes

$$\mathbf{P}(u) = \mathbf{P}_0(1-u)^3 + 3\mathbf{P}_1 u(1-u)^2 + 3\mathbf{P}_2 u^2(1-u) + \mathbf{P}_3 u^3, \qquad 0 \leq u \leq 1$$

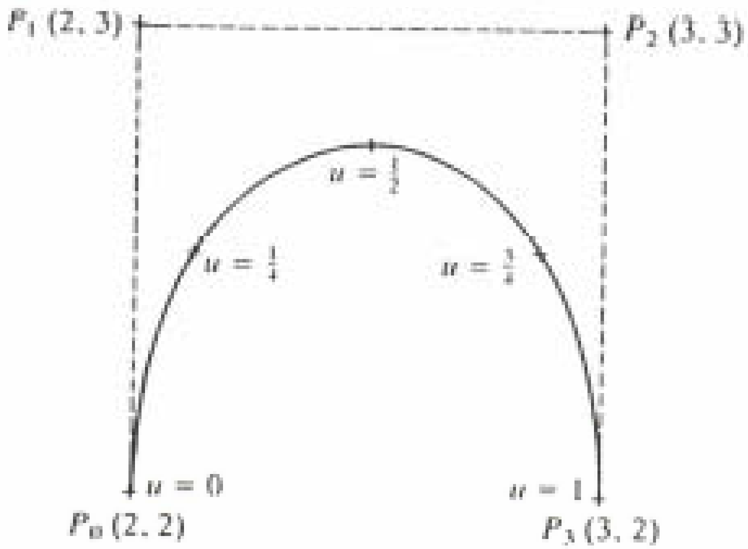



**FIGURE 5-49**
Bezier curve and generated points.

# Example Problem for Finding the Bezier Curve

$$\mathbf{P}(u) = \mathbf{P}_0(1-u)^3 + 3\mathbf{P}_1 u(1-u)^2 + 3\mathbf{P}_2 u^2(1-u) + \mathbf{P}_3 u^3, \qquad 0 \le u \le 1$$

$\mathbf{P}_0 = [2 \quad 2 \quad 0]^T$, $\mathbf{P}_1 = [2 \quad 3 \quad 0]^T$, $\mathbf{P}_2 = [3 \quad 3 \quad 0]^T$, and $\mathbf{P}_3 = [3 \quad 2 \quad 0]^T$

Substituting the $u$ values into this equation gives

$$\mathbf{P}(0) = \mathbf{P}_0 = [2 \quad 2 \quad 0]^T$$

$$\mathbf{P}\left(\frac{1}{4}\right) = \frac{27}{64}\mathbf{P}_0 + \frac{27}{64}\mathbf{P}_1 + \frac{9}{64}\mathbf{P}_2 + \frac{1}{64}\mathbf{P}_3 = [2.156 \quad 2.563 \quad 0]^T$$

$$\mathbf{P}\left(\frac{1}{2}\right) = \frac{1}{8}\mathbf{P}_0 + \frac{3}{8}\mathbf{P}_1 + \frac{3}{8}\mathbf{P}_2 + \frac{1}{8}\mathbf{P}_3 = [2.5 \quad 2.75 \quad 0]^T$$

$$\mathbf{P}\left(\frac{3}{4}\right) = \frac{1}{64}\mathbf{P}_0 + \frac{9}{64}\mathbf{P}_1 + \frac{27}{64}\mathbf{P}_2 + \frac{27}{64}\mathbf{P}_3 = [2.844 \quad 2.563 \quad 0]^T$$

$$\mathbf{P}(1) = \mathbf{P}_3 = [3 \quad 2 \quad 0]^T$$

Observe that $\sum_{i=0}^{3} B_{i,3}$ is always equal to unity for any $u$ value. Figure 5-49 shows the curve and the points.