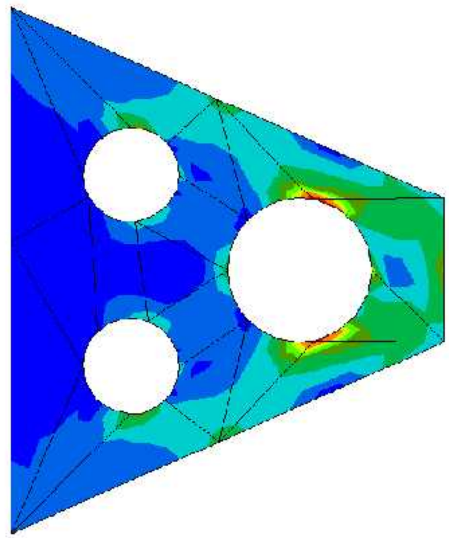
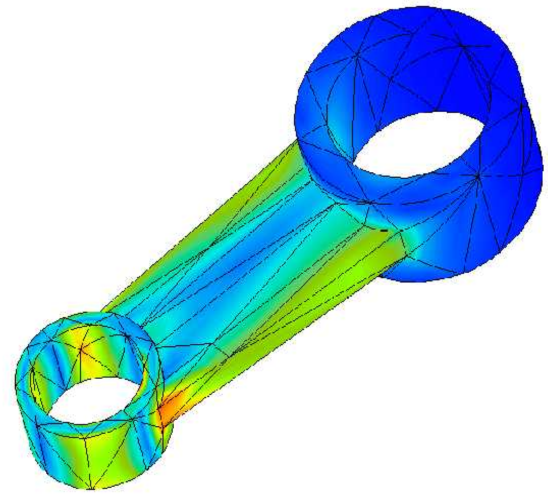
Introduction to Design Optimization

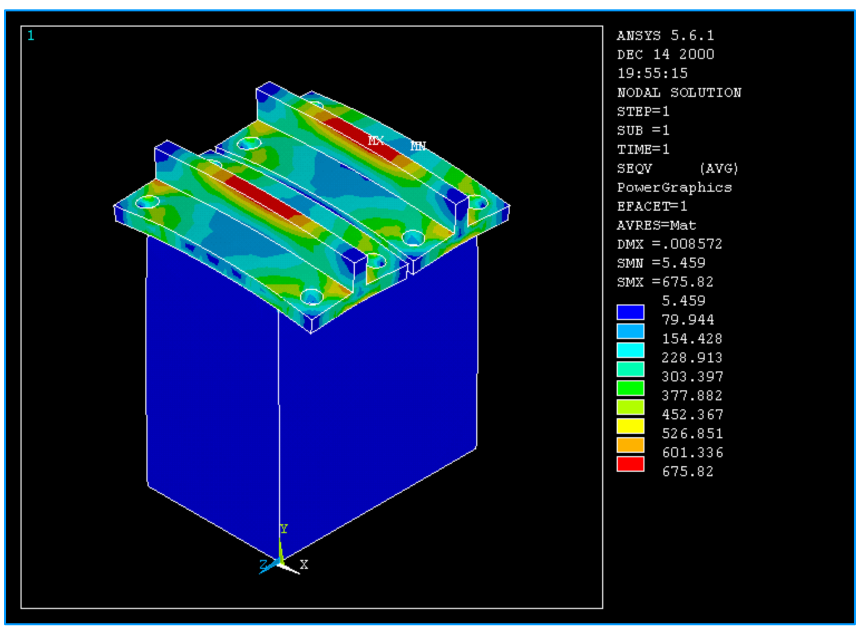
Examples and Constraint Penalty Functions

Various Design Objectives



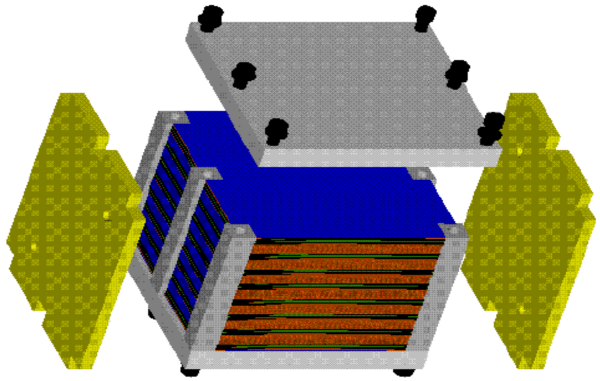


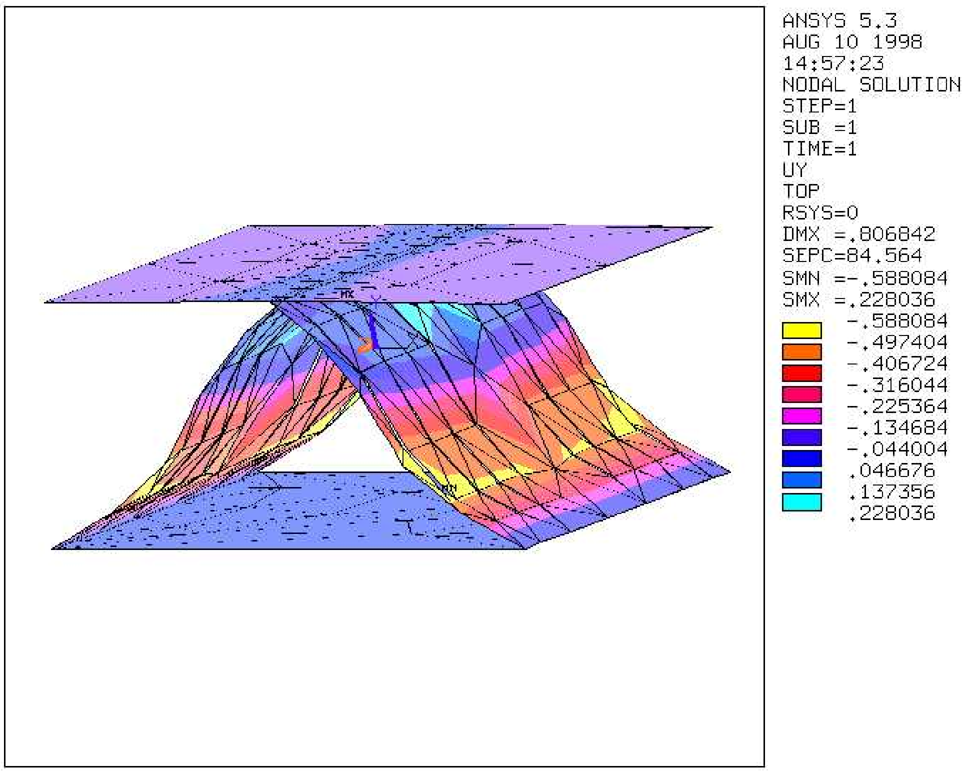
Minimum Weight
(under Allowable Stress)



A PEM Fuel Cell Stack with Even Compression over Active Area

(Minimum Stress Difference)

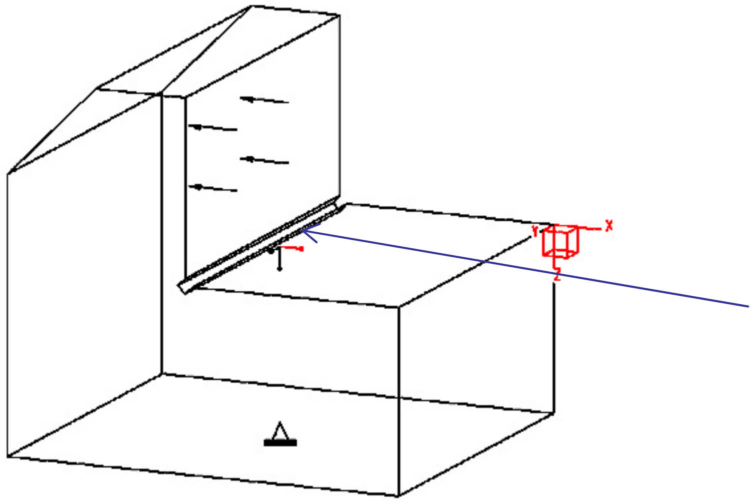




A PEM Fuel Cell Stack Multi-Functional Panel with Ideal Stiffness – to Accommodate Thermal- and Hydro-Expansions

(Minimum Difference between Ideal Stiffness and Calculated Stiffness):

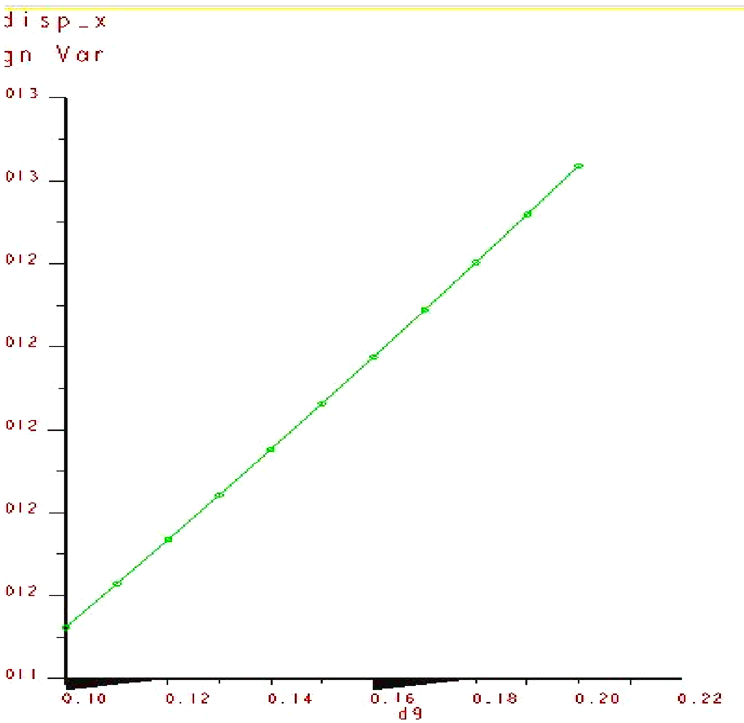
Find a panel design with the ideal stiffness.

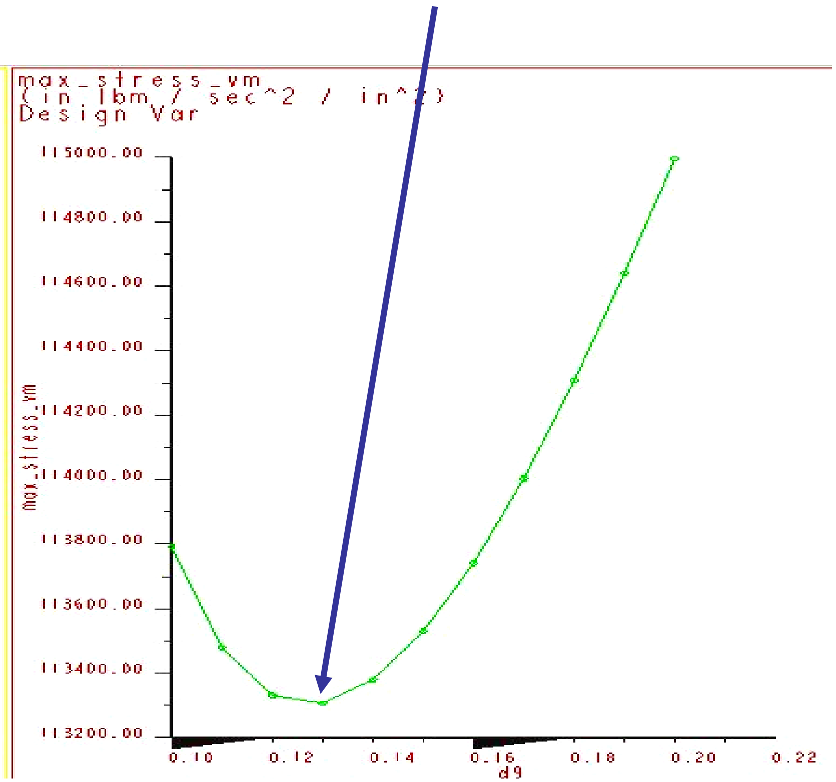


Minimum Maximum Stress in the Structure

Optimized Groove Dimension

to Avoid Stress Concentration or Weakening of the Structure





Engineering Applications of Optimization

- **Design** - determining design parameters that lead to the best "performance" of a mechanical structure, device, or system.

  "Core of engineering design, or the systematic approach to design" (Arora, 89)

- **Planning**
  - production planning - minimizing manufacturing costs
  - management of financial resources - obtaining maximum profits
  - task planning (robot, traffic flow) - achieving best performances
- **Control and Manufacturing** - identifying the optimal control parameters for the best performance (machining, trajectory, etc.)
- **Mathematical Modeling** - curve and surface fitting of given data with minimum error

Commonly used tool: OPT function in FEA; MATLAB Optimization Toolbox

What are common for an optimization problem?

- There are multiple solutions to the problem; and the optimal solution is to be identified.
- There exist one or more objectives to accomplish and a measure of how well these objectives are accomplished (measurable performance).
- Constraints of different forms (hard, soft) are imposed.
- There are several key influencing variables. The change of their values will influence (either improve or worsen) the “measurable performance” and the degree of violation of the “constraints.”

Solution Methods

- Optimization can provide either
  - a closed-form solution, or
  - a numerical solution.
- Numerical optimization systematically and efficiently adjusts the influencing variables to find the solution that has the best performance, satisfying given constraints.
- Frequently, the design objective, or cost function cannot be expressed in the form of simple algebra. Computer programs have to be used to carryout the evaluation on the design objective or costs. For a given design variable, α, the value of the objective function, $f(\alpha)$, can only be obtained using a numerical routine. In these cases, optimization can only be carried out numerically.

α → Computer Program (no simple algebra) → $f(\alpha)$

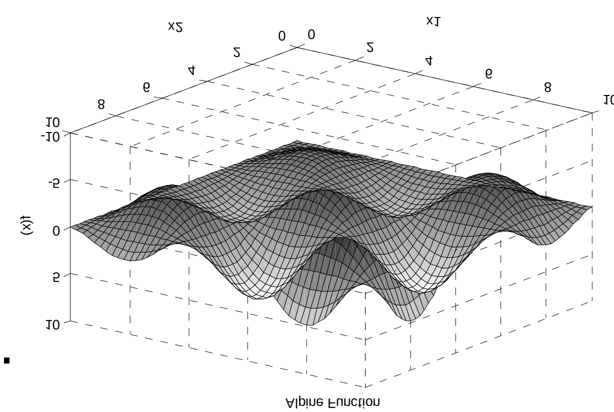
e.g. Minimize the maximum stress in a tents/tension structures using FEA.

Definition of Design Optimization

An optimization problem is a problem in which certain parameters (*design variables*) needed to be determined to achieve the best measurable performance (*objective function*) under given *constraints*.

Classification of the Optimization Problems

- **Type of design variables**
  - optimization of continuous variables
  - integer programming (discrete variables)
  - mixed variables
- **Relations among design variables**
  - nonlinear programming $\quad e.g.\ \ f(X) = Ae^{-x_1} + Bx_2$
  - linear programming $\quad e.g.\ \ f(X) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$
- **Type of optimization problems**
  - unconstrained optimization
  - constrained optimization
- **Capability of the search algorithm**
  - search for a local minimum
  - global optimization; multiple objectives; etc.



Automation and Integration

- Formulation of the optimization problems
  - specifying design objective(s)
  - specifying design constraints
  - identifying design variables
- Solution of the optimization problems
  - selecting appropriate search algorithm
  - determining start point, step size, stopping criteria
  - interpreting/verifying optimization results
- Integration with mechanical design and analysis
  - **black box** analysis functions serve as objective and constraint functions (e.g. FEA, CFD models)
  - incorporating optimization results into design

An Example Optimization Problem

Design of a thin wall tray with minimal material:

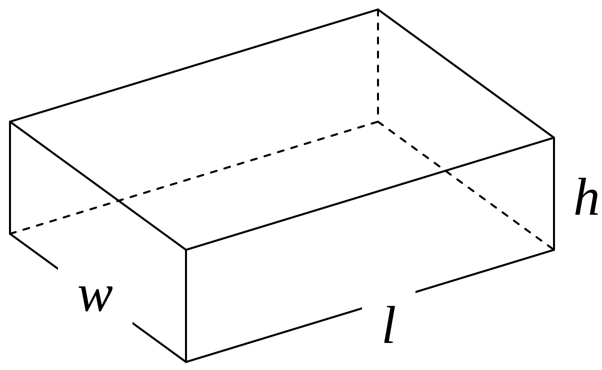
The tray has a specific volume, V, and a given height, H. The design problem is to select the length, l, and width, w, of the tray.

Given

$$lwh = V \qquad h = H$$

A “workable design”:

$$lw = \frac{V}{H}$$



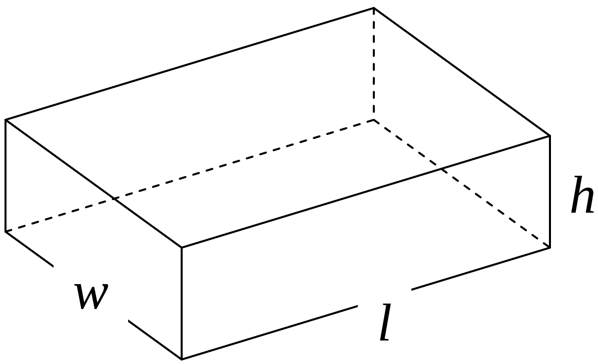
Pick either l or w and solve for others

An "Optimal Design"

- The design is to minimize material volume, (or weight), where "T" is an acceptable small value for wall thickness.

Minimize $$V_m(w,l,h) = T(\underbrace{wl}_{bottom} + \underbrace{2lh + 2wh}_{sides})$$

subject to $$\left.\begin{array}{c} lwh = V \\ h = H \\ l \geq 0 \\ w \geq 0 \end{array}\right\}$$ constraints (functions)



Design variables: w, l, and h.

Standard Mathematical Form

$$\min_{w.r.t.\, l,w,h} \quad T(wl + 2lh + 2wh)$$ - objective function

Subject to

$$\left.\begin{aligned} lwh - V &= 0 \\ h - H &= 0 \end{aligned}\right\}$$ - equality constraints

- inequality constrains

$$\left.\begin{aligned} -l &\le 0 \\ -w &\le 0 \end{aligned}\right\}$$ - variable bounds

$$\vec{x} = [l, w, h]^T$$ - design vector

- for use of any available optimization routines

Analytical (Closed Form) Solution

- Eliminate the equality constrains, convert the original problem into a single variable problem, then solve it.

from $h = H$ & $l\,w\,H = V;$ solve for l: $l = \dfrac{V}{Hw}$

thus

$$\min_{w} T\left(\frac{V}{Hw}w + 2\frac{V}{Hw}H + 2wH\right) \longrightarrow \min_{w} T\left(\frac{V}{H} + 2\frac{V}{w} + 2wH\right) = f(w)$$

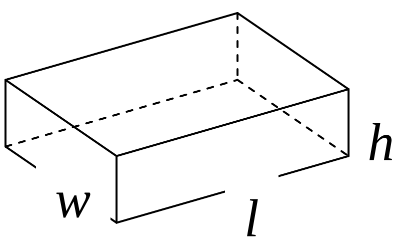
from $\dfrac{df(w)}{dw} = 0,$ we have $w^2 = \dfrac{V}{H},$ then the design optimum $w^* = \sqrt{\dfrac{V}{H}}$

- a stationary point

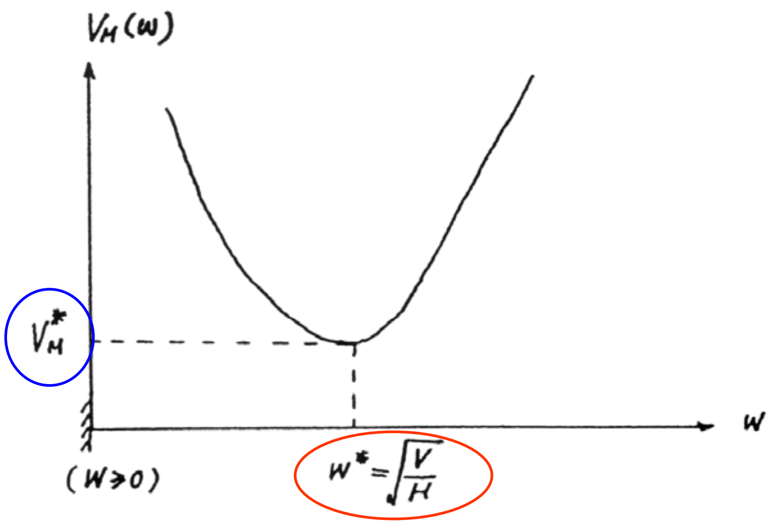
- Discard the negative value, since the inequality constraint is violated.
- The optimal value for l:

$$l^* = \frac{V}{Hw^*} = \sqrt{\frac{V}{H}} = w^*$$

$$V_M^* = T\left(\frac{V}{H} + 2hw^* + \frac{2V}{w^*}\right) = T\left(\frac{V}{H} + 4\sqrt{VH}\right)$$



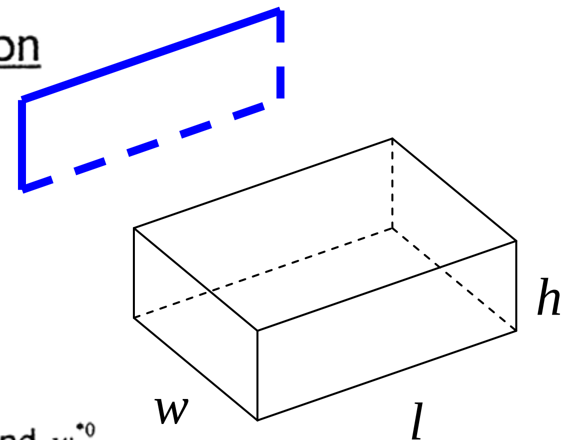
Graphical Solution



$w \leq \infty$ or $l \geq 0$

no width & length limitations
no violated constraints.

<u>Change of Constraints and Their Influence to the Final Solution</u>



Consider a modified problem:

$$\min_{l,w,h} \quad V_m = T\,(w \times l + 2 \times l \times h + 2 \times w \times h)$$

s.t.

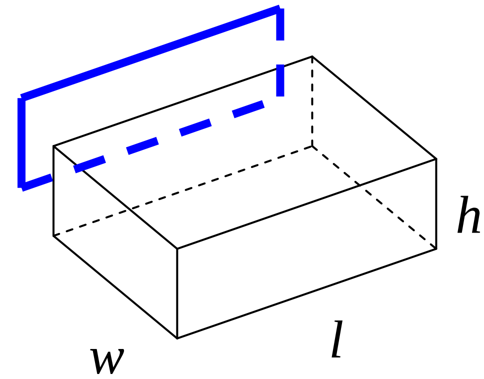
$$lwh = V$$
$$h = H$$
$$w \geq 0$$
$$l \geq 0$$

Handled as an unconstrained problem and found w^{*0}

$\boxed{w \leq W \quad \text{maximum width / add a new constraint}}$

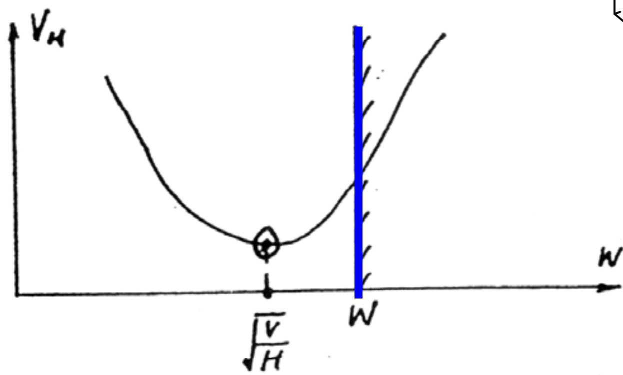
Follow the previous example:

unconstrained optimum: $w^{*0} = \sqrt{\frac{V}{H}}$



- For $W \geq w^{*0} = \sqrt{\frac{V}{H}}$

  $w^* = l^* = \sqrt{\frac{V}{H}}$

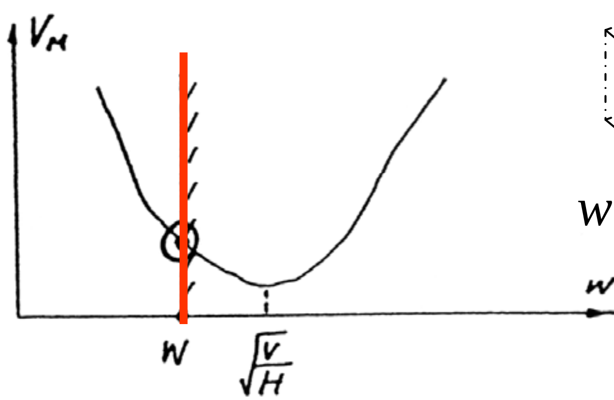


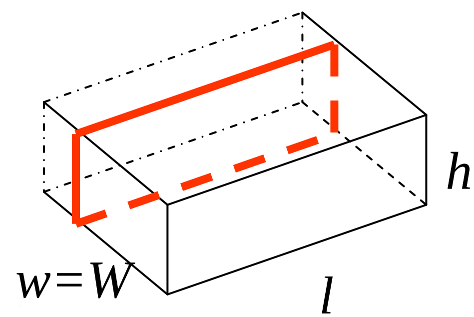
The constrained optimum is not changed, no active constraints.

- For $W \leq w^{*0} = \sqrt{\frac{V}{H}}$

  $w^* = W \neq w^{*0}$

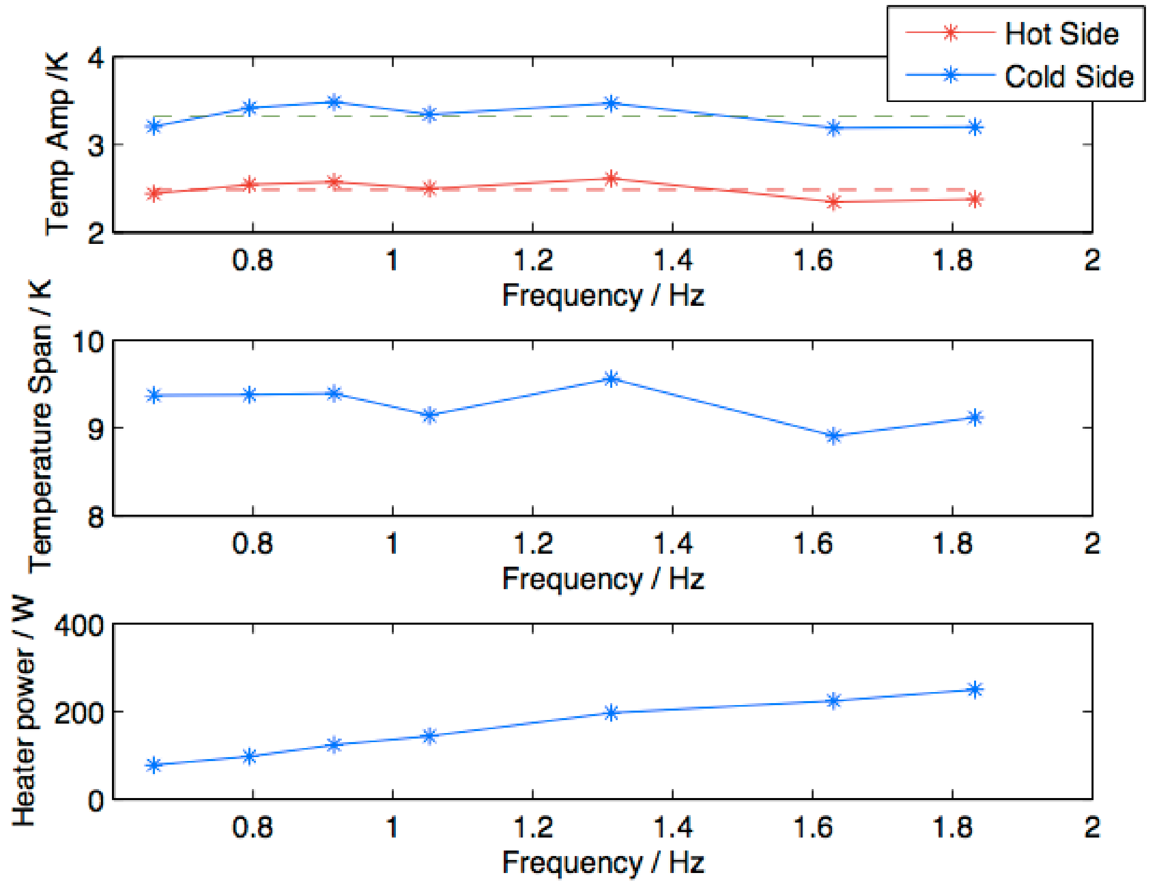
  $l^* = \frac{V}{HW}$





Constraint $w \leq W$ is "active."

Regenerator Convection Heat Transfer Coefficient



Regenerator Convection Heat Transfer Coefficient

Using harmonic approximation techniques one can show that the temperature amplitude T_{f1} and the time averaged temperature gradient $\mathrm{d}T_{f0}/\mathrm{d}x$ are correlated through

$$T_{f1} = G_f(f)\frac{\mathrm{d}T_{f0}}{\mathrm{d}x} \tag{1}$$

Where G_f is the fluid transfer function and carries operational, geometrical and material information. The HTC is embedded in G_f.

$$G_f(\omega) = -\frac{(C + i\omega)}{-\omega^2 + (B + C)i\omega} L_{\mathrm{s,eff}} \frac{\omega}{2\epsilon} \tag{2}$$

where $B = ha_{\mathrm{sf}}/(\epsilon(\rho c)_f)$ and $C = ha_{\mathrm{sf}}/((1 - \epsilon)(\rho c)_s)$. Using equation (1) we can use measured temperature amplitudes and gradients to correlate them through G_f. The only degree of freedom is the HTC h. We denote measured temperature amplitudes at frequencies f_i with $T^i_{\mathrm{f1}}(f_i)$. The HTC can be found by minimizing the residual (objective function $O(h)$) with respect to h

$$\min_h O(h) = \min_h \sum_{i=1}^{n} \left(T^i_{\mathrm{f1}}(f_i) - G_f(f_i, h)\frac{\mathrm{d}T_{\mathrm{f0}}}{\mathrm{d}x} \right)^2 \tag{3}$$

Regenerator Convection Heat Transfer Coefficient

If we assume the heat transfer coefficient to be constant (independent of frequency) we get a model output as shown below

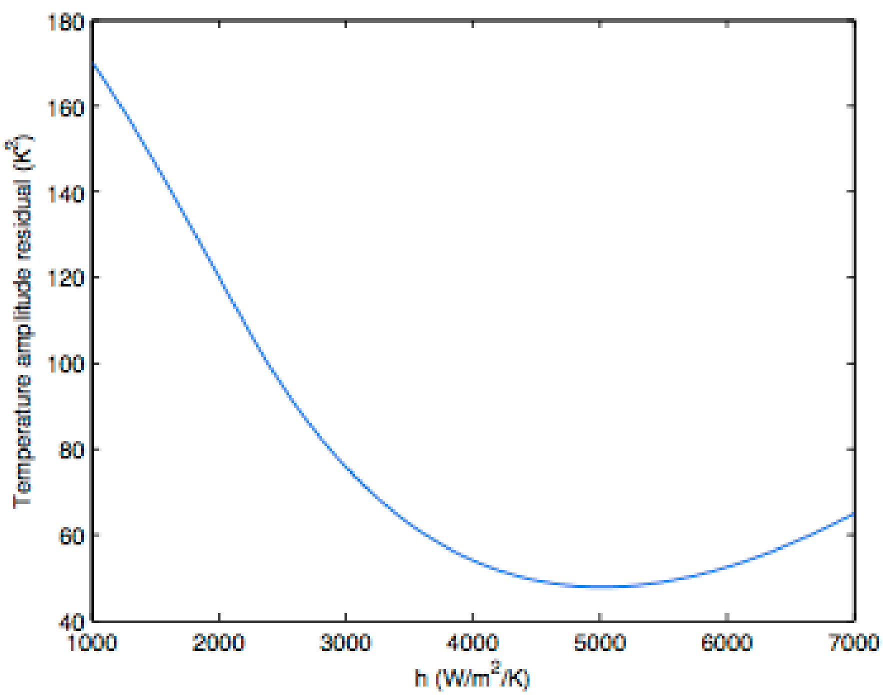


Figure 2: Temperature amplitude residual $O(h)$ as function of h

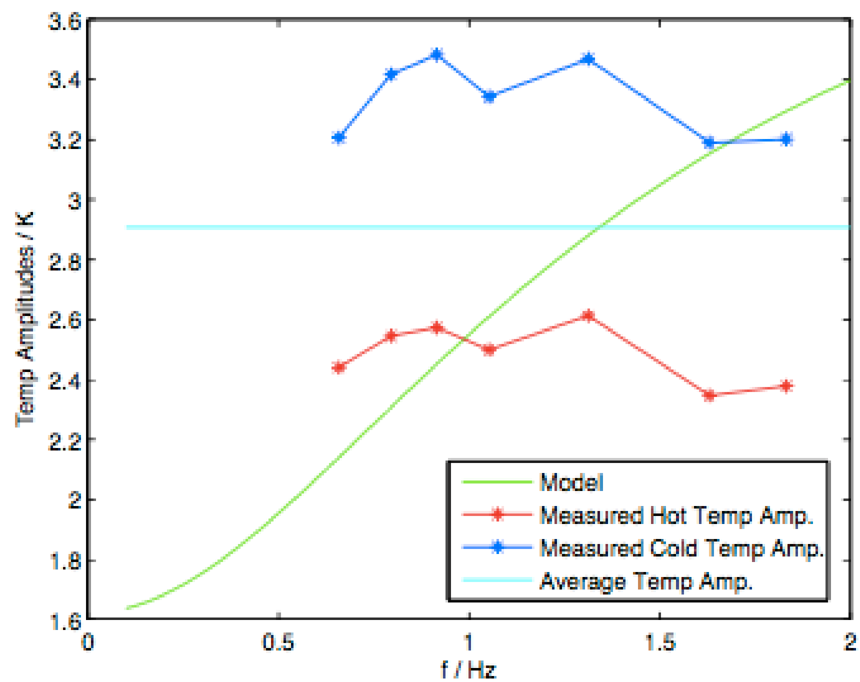


Figure 3: Model response using h=5000 W/m2/K compared to experimental data

In figure (2) a clear minima is observable around $h = 5000\,\mathrm{W/m^2/K}$. The model response using $h = 5000\,\mathrm{W/m^2/K}$ compared to the experimental values is shown in figure (3). The model response using constant HTC is poor, large deviations from experiment and model are observable.

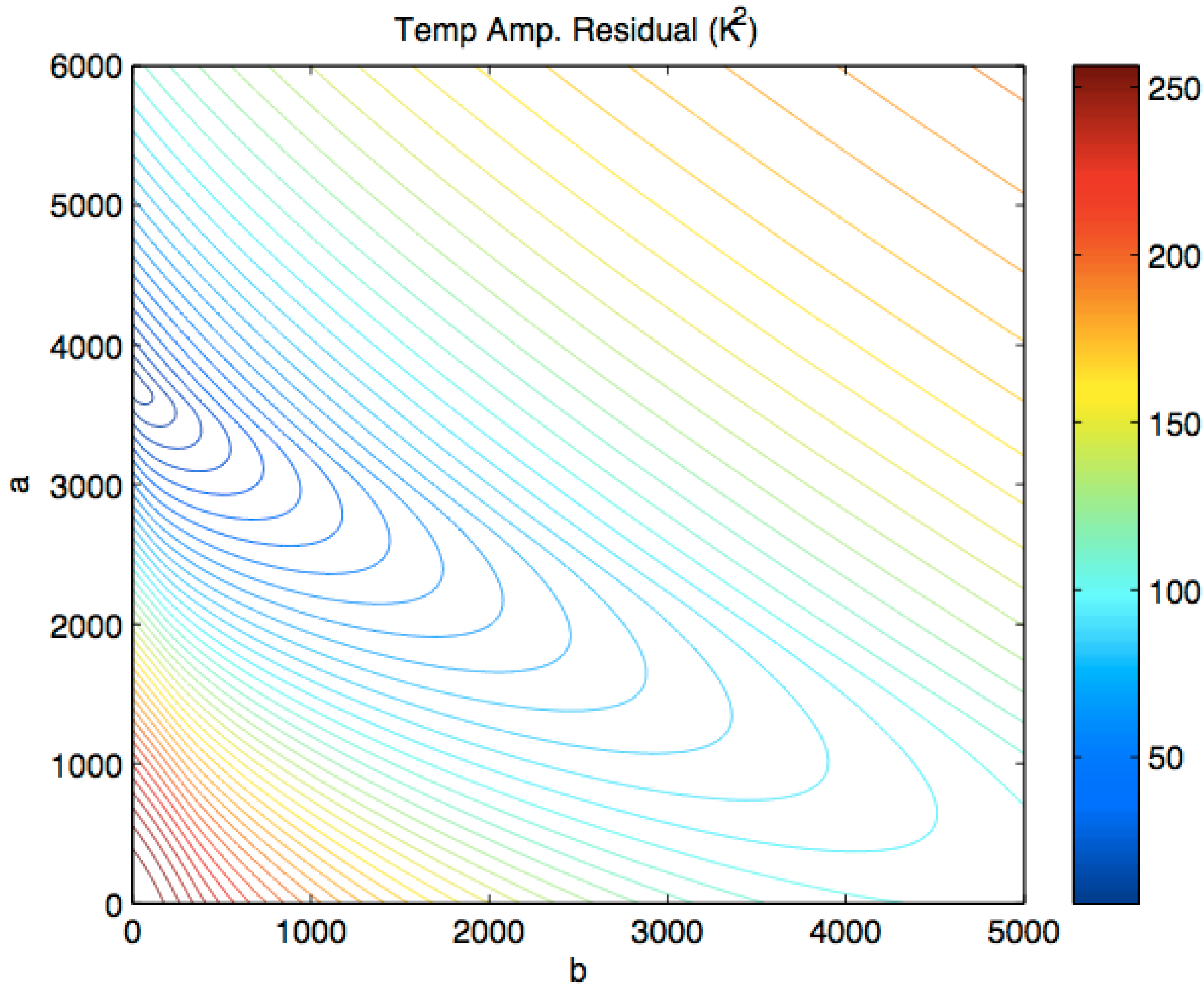
Regenerator Convection Heat Transfer Coefficient

In this section we make an attempt to improve the model quality by choosing a specific non constant form of the heat transfer coefficient, that is,

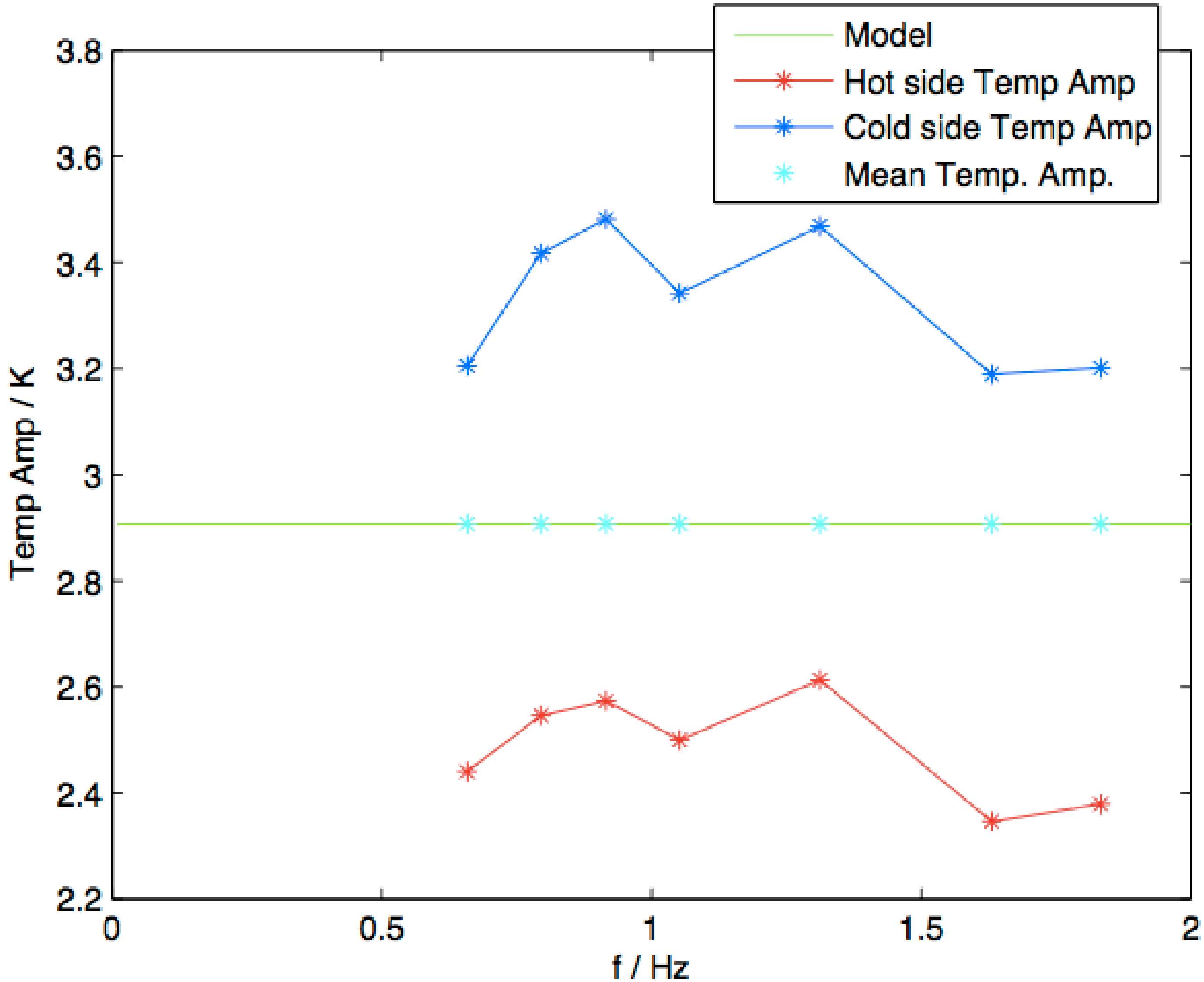
$$h(f) = b + a \cdot f \tag{4}$$

The minima of the residual $O(h)$ must now be discussed in terms of a and b. Figure (4) shows a clear minima for $b = 0$ and $a = 3730$. Although this model will show a much smaller error we must argue that $b = 0$ is rather not physical since heat transfer can still occur, even when $\mathrm{Re} = 0$. The model response is shown in figure (5). The mean temperature amplitudes and model coincide perfectly.

Regenerator Convection Heat Transfer Coefficient



Regenerator Convection Heat Transfer Coefficient



Procedures for Solving an Eng. Optimization Problem

- **Formulation of the Optimization Problem**
  - Simplifying the physical problem
  - identifying the major factor(s) that determine the performance or outcome of the physical system, such as costs, weight, power output, etc. – objective
  - Finding the primary parameters that determine the above major factors - the design variables
  - Modeling the relations between design variables and the identified major factor - objective function
  - Identifying any constraints imposed on the design variables and modeling their relationship – constraint functions
- **Selecting the most suitable optimization technique or algorithm to solve the formulated optimization problem.**

  - requiring an in-depth know-how of various optimization techniques.
- **Determining search control parameters**

  - determining the initial points, step size, and stopping criteria of the numerical optimization
- **Analyzing, interpreting, and validating the calculated results**

  An optimization program does not guarantee a correct answer, one needs to
  - prove the result mathematically.
  - verify the result using check points.

Standard Form for Using Software Tools for Optimization (e.g. MatLab Optimization Tool Box)

Denoting the optimization variables *X*, as a n-dimensional vector, where the n variables are its componets, and the objective function *F(X)* we search for :

$$\mathbf{X}^* \in R^n \text{ so that } F(\mathbf{X}^*) = \min F(\mathbf{X})$$

subject to

$$\mathbf{X}_l \le \mathbf{X}^* \le \mathbf{X}_u$$

Regional constraints

$$G_i(\mathbf{X}^*) \ge 0 \qquad i = 1, 2, \ldots, m$$

and

Behavior constraints

$$H_j(\mathbf{X}^*) = 0 \qquad j = 1, 2, \ldots, q$$

Where *m* are the number of inequality constraints and *q* the number of equality constraints

Use of MATLAB
Optimization Toolbox

Notes

- A maximization problem can be converted into a minimization problem by:

$$\max \quad f(\vec{x}) \Rightarrow \min \frac{1}{f(\vec{x})} \quad \text{or} \quad \min\left\{-f(\vec{x})\right\}$$

- $\varepsilon_g 4$ can be converted into $\varepsilon_g 3$:

$$x_i - x_i^{(u)} \le 0$$
$$x_i^{(l)} - x_i \le 0$$

  adding *2N* inequality constraints

Assignment: (notebook only, not to turn in)

- Consider a circular tray, find the minimum v_m with $h = H$, and also with h free. The diameter of the tray, d, is a design variable.

- Compare the above two "competing" design in terms of v_m.

Geometric Interpretation of the Objective Function

- The Objective function can be interpreted to be a surface of dimension *n* embedded in a space of dimension *n+1*. This is easy to visualize for a 2 parameter problem.

- The optimization process can be compared to “mountain climbing in a dense fog, having as only tool an altimeter”.

Treatment of Constraints

- Equality constraints effectively reduce the dimensions of the design space by 1.

- Inequality constraints can be mathematically enforced by the introduction of ***penalty functions***, so that a large value is added to the function when the constraints are violated. We can define the penalty function ***P(X)***:

$$P(\mathbf{X}) = \begin{cases} 0 & \text{for } \mathbf{X} \in R_f^n \\ +\infty & \text{for } \mathbf{X} \notin R_f^n \end{cases}$$

where R_f^n is the subset of R^n corresponding to the feasible design

And the minimization process can be extended to the augmented function ***D(X)***:

$$D(\mathbf{X}) = F(\mathbf{X}) + P(\mathbf{X})$$

Exterior Penalty Functions

- Exterior penalty function is $\mathbf{X} \notin R_f^n$ for

Defining penalty function as: $S(X) = \sum_i \delta_i \left|G_i(\mathbf{X})\right|^\alpha + \sum_j \left|H_j(\mathbf{X})\right|^\beta$

where

$$\delta_j = \begin{cases} 0 & \text{if } G_j(\mathbf{X}) \geq \mathbf{0} \\ 1 & \text{if } G_j(\mathbf{X}) < \mathbf{0} \end{cases}$$

$$S(X) = 0 \quad \text{if } \mathbf{X} \in R_f^n$$

and

$$S(X) > 0 \quad \text{if } \mathbf{X} \notin R_f^n$$

and α and β are constants, usually having values of 1 or 2, and the functions G_i and H_j are those in Equations (9.3) and (9.4). Note that

Exterior Penalty Functions

For any positive number ρ, we can define the augmented objective function as

$$D(\mathbf{X}, \rho) = F(\mathbf{X}) + \frac{1}{\rho} S(\mathbf{X})$$

and observe that $D(\mathbf{X}, \rho) = F(\mathbf{X})$ if and only if $\mathbf{X}$ is in the feasible region; otherwise $D(\mathbf{X}, \rho) > F(\mathbf{X})$. The $S(\mathbf{X})/\rho$ term approximates the discontinuous penalty function $P(\mathbf{X})$ in Equation as $\rho \to 0$. Thus the exterior penalty function method consists of solving a sequence of unconstrained optimizations for $k = 0, 1, 2, \ldots$, given by

$$\min D(\mathbf{X}, \rho_k) = \min\left[F(\mathbf{X}) + \frac{1}{\rho_k}\left(\sum_i \delta_i \left|G_i(\mathbf{X})\right|^{\alpha} + \sum_j \left|H_j(\mathbf{X})\right|^{\beta} \right)\right]$$

using a strictly decreasing sequence of positive numbers ρ_k. The optimal values $\mathbf{X}_k$ for ρ_k will converge to the real optimal values $\mathbf{X}^*$ as k increases and ρ_k approaches 0.

Exterior Penalty Functions - Example

Find the minimum of

$$F(x) = x^2 \quad (x \in R)$$

subject to the constraint $x - 1 \geq 0$. The optimal solution clearly is $x^* = 1$, so we need to show that the solution obtained by the exterior penalty method converges to this solution.

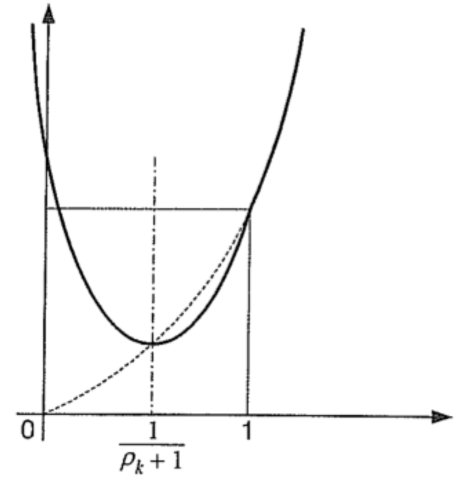
ANSWER

Let's form the augmented objective function,
Then we have the unconstrained optimization problem

$$\min D(x, \rho_k) = \min\left[x^2 + \frac{1}{\rho_k}\delta(x-1)^2\right]$$

Here δ has the value of 1 for $x < 1$ and 0 otherwise.

Exterior Penalty Functions Example



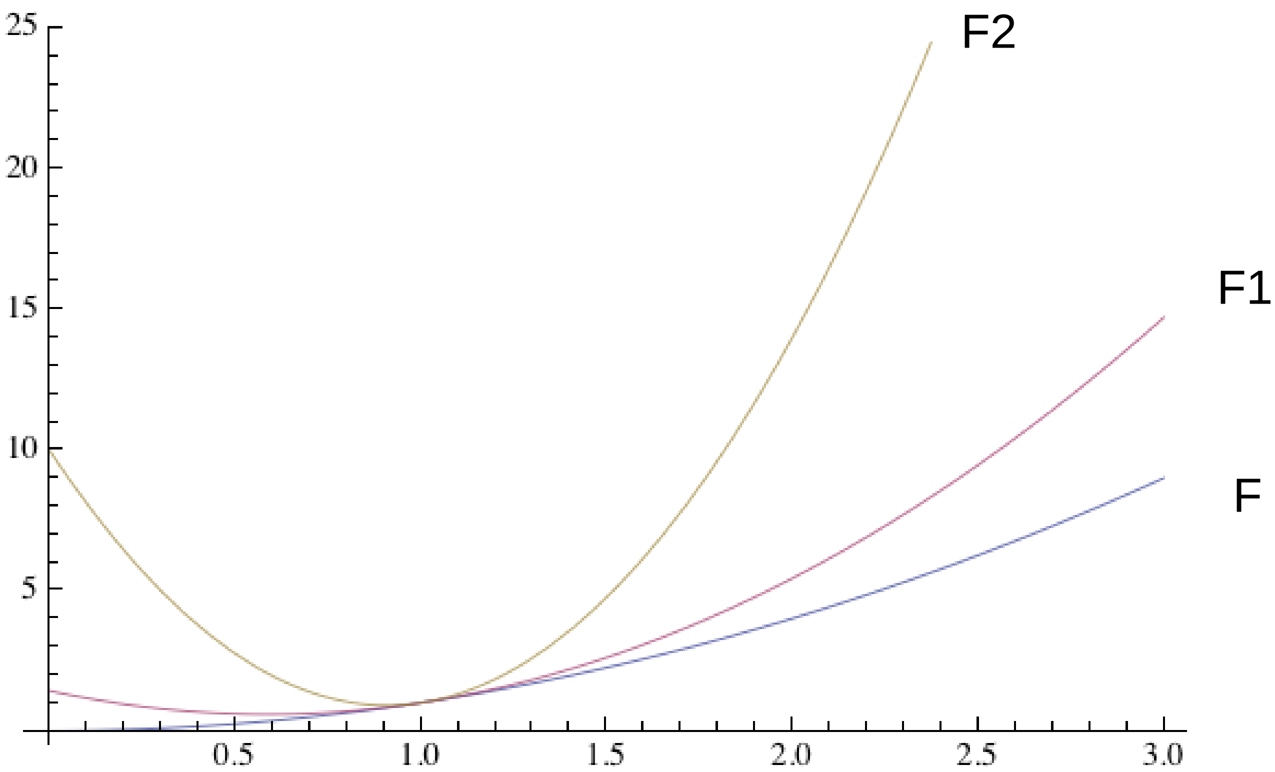
For any positive ρ_k, the function D is convex downward, as shown in Figure and its minimum is at the point

$$x_k = \frac{1}{\rho_k + 1}$$

Note that, for every positive ρ_k, this point is infeasible for the original problem because it is smaller than 1. As ρ_k approaches 0, points x_k approach $x = 1$ from outside the feasible region.

Exterior Penalty Functions Example

```
F = x^2;
F1 = x^2 + (1 / .7) * (x - 1) ^2;
F2 = x^2 + (1 / .1) * (x - 1) ^2;
Plot[{F, F1, F2}, {x, 0, 3}]
```



Interior Penalty Functions

- Interior penalty function is applied f if $\mathbf{X} \in R_f^n$

Consider therefore the optimization problem

$$\min F(\mathbf{X})$$

subject to

$$G_i(\mathbf{X}) \geq 0 \qquad i = 1, 2, \ldots, m$$

A good choice for the barrier function that will provide the walls at the boundaries of the feasible region would be

$$B(\mathbf{X}) = \frac{1}{G_i(\mathbf{X})}$$

The augmented function ***D(X,ρ)***:

$$D(\mathbf{X}, \rho) = F(\mathbf{X}) + \rho B(\mathbf{X})$$

Interior Penalty Functions

Given the augmented function $D(\mathbf{X}, \rho) = F(\mathbf{X}) + \rho B(\mathbf{X})$

where ρ is a positive number. Similarly, the interior penalty method consists of solving a sequence of unconstrained optimizations for $k = 0, 1, 2, \ldots,$ given by

$$\min D(\mathbf{X}, \rho_k) = \min\left[F(\mathbf{X}) + \rho_k \sum_i \frac{1}{G_i(\mathbf{X})}\right]$$

using a strictly decreasing sequence of positive numbers ρ_k. The optimal values $\mathbf{X}_k$ for ρ_k will converge to the real optimal values $\mathbf{X}^*$ as k is increased and ρ_k approaches 0.

Interior Penalty Functions - Example

Find the minimum of

$$F(x) = \frac{1}{2}x \qquad (x \in R)$$

subject to the constraint $x - 1 \geq 0$. The optimal solution clearly is $x^* = 1$, so we need to show that the solution obtained by the interior penalty method converges to this solution.

ANSWER

Let's form the augmented objective function

$$\min D(x, \rho_k) = \min\left[\frac{1}{2}x + \rho_k \frac{1}{x-1}\right]$$

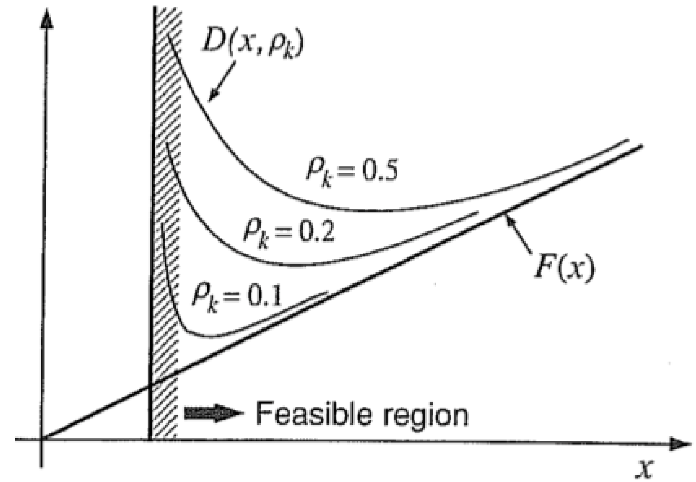
Interior Penalty Functions - Example

Hence the unconstrained minimum of D is[2]

$$x_k = 1 + \sqrt{2\rho_k}$$

and the minimum value of D is

$$F(x_k) = \frac{1}{2} + \sqrt{\frac{\rho_k}{2}}$$



Note that, for every positive ρ_k, the optimal point is in the feasible region for the original problem because it is greater than 1. As ρ_k approaches 0, the points x_k approach $x = 1$. The original and the augmented objective functions for a few values of ρ_k are illustrated in Figure When the procedure described is carried out by use of a numerical algorithm, the initial value for the search has to be chosen in the feasible region.