§5.1 Two-Dimensional Arbitrarily Oriented Beam Element.

*Frames* and *grids* are planar and spatial assemblies of beam elements respectively.

- Frames: structure with external loads applied within the plane of the structure.
- Grids: structures with external loads applied primarily out of the structure’s plane.
Lecture 11: Frame and Grid Elements.

- When dealing with such assemblies the major difficulty is dealing with the various element orientations.
- Frames:
  - Orientation changes are contained within the $xy$ plane (rotations are about the $z$ axis).
- Grids:
  - Orientation changes are out of the $xy$ plane (rotations are about the $y$ axis).
- We’ll look at using the results of §3.7 (spatial rotations) to construct both the frame/grid element equations.
- First, we will add some capabilities to our standard beam element of Chapter 4.
Lecture 11: Frame and Grid Elements.

- The standard beam problem:

  When oriented arbitrarily in the $xy$ plane, a beam element could be subject to axial loads.

  \[
  \begin{bmatrix}
  -1 & 1 \\
  1 & -1
  \end{bmatrix}
  \begin{bmatrix}
  \hat{d}_{1x} \\
  \hat{d}_{2x}
  \end{bmatrix}
  =
  \begin{bmatrix}
  \hat{f}_{1x} \\
  \hat{f}_{2x}
  \end{bmatrix}
\]

- No need to model (capture) the axial stiffness of the actual structure.

- When oriented arbitrarily in the $xy$ plane, a beam element could be subject to axial loads.

  Transverse deflections and bending.

  \[
  \begin{bmatrix}
  \hat{f}_{1y} \\
  \hat{f}_{2y} \\
  \hat{m}_{1z}
  \end{bmatrix}
  =
  \begin{bmatrix}
  12 & 6L & -12 & 6L \\
  6L & 4L^2 & -6L & 2L^2 \\
  -12 & -6L & 12 & -6L
  \end{bmatrix}
  \begin{bmatrix}
  \hat{d}_{1y} \\
  \hat{d}_{2y} \\
  \hat{\phi}_1
  \end{bmatrix}
\]

  \[
  \begin{bmatrix}
  \hat{f}_{1y} \\
  \hat{f}_{2y} \\
  \hat{m}_{2z}
  \end{bmatrix}
  =
  \begin{bmatrix}
  12 & 6L & -12 & 6L \\
  4L^2 & -6L & 2L^2 & 12 \\
  -12 & -6L & 12 & -6L
  \end{bmatrix}
  \begin{bmatrix}
  \hat{d}_{1y} \\
  \hat{d}_{2y} \\
  \hat{\phi}_1
  \end{bmatrix}
\]

  Superpose *(Small Deflections)*

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Principle of superposition can be invoked when the deformations are small.

Assume no coupling between the axial and transverse mechanics.
Lecture 11: Frame and Grid Elements.

\[
\begin{pmatrix}
\hat{f}_{1x} \\
\hat{f}_{1y} \\
\hat{m}_{1z} \\
\hat{f}_{2x} \\
\hat{f}_{2y} \\
\hat{m}_{2z}
\end{pmatrix} = 
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\begin{pmatrix}
\hat{d}_{1x} \\
\hat{d}_{1y} \\
\phi_1 \\
\hat{d}_{2x} \\
\hat{d}_{2y} \\
\phi_2
\end{pmatrix}
\]

Logan:

\[
C_1 = \frac{AE}{L}, \quad C_2 = \frac{EI}{L^3}
\]

The frame element equations.

- §5.1 goes on to put the frame element equations in terms of global coordinates.
- Before doing this, we will look at the grid element equations.
§5.4 Grid Equations.

For *grid* elements, out of plane loading can lead to twist…

Components of a moment vector – sense governed by ‘RHR’.
Lecture 11: Frame and Grid Elements.

- We need to develop a FE model for the twist of a beam.
- This is a new “twist element” that is created following the first 4 of Logan’s steps.
- Step 1: Set the Element Type.
  - We define the applied moments and nodal displacements that will be related through a stiffness matrix.
  - Note that the applied (or external) loads and the displacements are defined by a coordinate system.
  - The internal loads are governed by an independent sign convention.

\[ \hat{m}_1 = -\hat{m}_x \]
\[ \hat{m}_2 = +\hat{m}_x \]
Lecture 11: Frame and Grid Elements.

- Step 2: Define the displacement function.
  - In the DSM everything is put in terms of an assumed displacement field.
  - The “displacement” in this case is the axial rotation, $\hat{\phi}_x$.

\[
\hat{\phi}_x = a_1 + a_2 \hat{x}
\]

\[
\begin{align*}
\hat{\phi}_x (0) &= \hat{\phi}_{1x} = a_1 \\
\hat{\phi}_x (L) &= \hat{\phi}_{2x} = a_1 + a_2 L
\end{align*}
\]

\[
\hat{\phi}_x = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} \hat{\phi}_{1x} \\ \hat{\phi}_{2x} \end{bmatrix}
\]

\[
\begin{align*}
N_1 &= 1 - \frac{\hat{x}}{L} \\
N_2 &= \frac{\hat{x}}{L}
\end{align*}
\]

The linear approximation to the displacement field produces the same shape functions as in the truss element development.
Step 3: Define the stress-strain relationships.

Once again we relate loads to displacements by first looking at how the (shear) stress is related to the (shear) strain.

Subbing our assumed displacement field into these stress strain relations gives us the terms of the stiffness matrix.

\[ d\hat{x} \cdot \gamma_{\text{max}} = R \cdot d\hat{\phi} \]

\[ \gamma_{\text{max}} = R \frac{d\hat{\phi}}{d\hat{x}} \]

\[ \gamma_{\text{max}} = R \left( \frac{\hat{\phi}_{2x} - \hat{\phi}_{1x}}{L} \right) \]

\[ \hat{m}_x = \int r \left( \frac{r}{R} \tau_{\text{max}} \right) dA \]

\[ \hat{m}_x = \frac{(\tau_{\text{max}})}{R} J, \quad J = \int r^2 dA \]

\[ \tau_{\text{max}} = G\gamma_{\text{max}} \]

\[ \hat{m}_x = \frac{GJ}{L} \left( \hat{\phi}_{2x} - \hat{\phi}_{1x} \right) \]
Lecture 11: Frame and Grid Elements.

- Step 4: Derive the element equations.

\[
\hat{m}_{1x} = -\hat{m}_x = \frac{-GJ}{L} (\hat{\phi}_{2x} - \hat{\phi}_{1x}) \\
\hat{m}_{2x} = +\hat{m}_x = \frac{+GJ}{L} (\hat{\phi}_{2x} - \hat{\phi}_{1x})
\]

\[
\begin{bmatrix}
\hat{f}_{1y} \\
\hat{m}_{1x} \\
\hat{m}_{1z} \\
\hat{f}_{2y} \\
\hat{m}_{2x} \\
\hat{m}_{2z}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\hat{d}_{1y} \\
\hat{\phi}_{1x} \\
\hat{\phi}_{1z} \\
\hat{d}_{2y} \\
\hat{\phi}_{2x} \\
\hat{\phi}_{2z}
\end{bmatrix} + \begin{bmatrix}
12 \frac{EI}{L^3} & 0 & 6 \frac{EI}{L^2} & -12 \frac{EI}{L^3} & 0 & 6 \frac{EI}{L^2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
6 \frac{EI}{L^2} & 0 & 4 \frac{EI}{L} & -6 \frac{EI}{L^2} & 0 & 2 \frac{EI}{L} \\
-12 \frac{EI}{L^3} & 0 & -6 \frac{EI}{L^2} & 12 \frac{EI}{L^3} & 0 & -6 \frac{EI}{L^2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
6 \frac{EI}{L^2} & 0 & 2 \frac{EI}{L} & -6 \frac{EI}{L^2} & 0 & 4 \frac{EI}{L}
\end{bmatrix} \begin{bmatrix}
\hat{d}_{1y} \\
\hat{\phi}_{1x} \\
\hat{\phi}_{1z} \\
\hat{d}_{2y} \\
\hat{\phi}_{2x} \\
\hat{\phi}_{2z}
\end{bmatrix}
\]
Lecture 11: Frame and Grid Elements.

The completed grid element equations are given by Eq. (5.4.16):

\[
\begin{bmatrix}
\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\
0 & \frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 \\
\frac{6EI}{L^2} & 0 & \frac{4EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\
0 & -\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 \\
\frac{6EI}{L^2} & 0 & \frac{2EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L}
\end{bmatrix}
\begin{bmatrix}
\hat{f}_{1y} \\
\hat{m}_{1x} \\
\hat{m}_{1z} \\
\hat{f}_{2y} \\
\hat{m}_{2x} \\
\hat{m}_{2z}
\end{bmatrix}
= 
\begin{bmatrix}
\hat{a}_{1y} \\
\hat{\phi}_{1x} \\
\hat{\phi}_{1z} \\
\hat{a}_{2y} \\
\hat{\phi}_{2x} \\
\hat{\phi}_{2z}
\end{bmatrix}
\]
Lecture 11: Frame and Grid Elements.

- Pg. # 242: equivalent 2\textsuperscript{nd} polar moment of area (‘\(J\)’) for rectangular cross sections.
- SC = “shear center”
- Both open and closed cross sections can be used within our beam element formulation.

**NOTE:**
- Recovering the torsional shear stress requires special procedures in these cases.
- Can not blindly apply stress-strain equations for the annular \(X\)-section.

<table>
<thead>
<tr>
<th>Cross Section</th>
<th>Torsional Constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Channel</td>
<td>(J = \frac{t^3}{3}(h+2b))</td>
</tr>
<tr>
<td></td>
<td>(e = \frac{h^2b^2b}{4I})</td>
</tr>
<tr>
<td>2. Angle</td>
<td>(J = \frac{1}{3}(b_1t_1^3 + b_2t_2^3))</td>
</tr>
<tr>
<td>3. Z section</td>
<td>(J = \frac{t^3}{3}(2b + h))</td>
</tr>
<tr>
<td>4. Wide-flanged beam with unequal flanges</td>
<td>(J = \frac{1}{3}(b_1t_1^3 + b_2t_2^3 + ht_0^3))</td>
</tr>
<tr>
<td>5. Solid circular</td>
<td></td>
</tr>
</tbody>
</table>
§5.1 and §5.4 Revisited.

- Frame and grid elements must be assembled in terms of a global frame of reference.
- For frame problems the structural components are within the $xy$ plane of the global frame (by convention).
  - We need to be able to rotate the global frame about the $z$ axis to align with each elemental frame.
- For grid problems, the grid elements will be within the $xz$ plane of the analysis (by convention).
  - We need to be able to rotate the global frame about the $y$ axis to align with each elemental frame.
- Consider rotations within the $z$ axis (as occurs with frames).
The rotation matrix is composed of nine direction cosines.

For rotation about the z axis, it is simple to define these direction cosines in terms of the angle of rotation.

\[
\begin{bmatrix}
\{d_{ix}\} & \{d_{iy}\} & \{d_{iz}\}
\end{bmatrix} = 
\begin{bmatrix}
\hat{i} \cdot \hat{i} & \hat{j} \cdot \hat{i} & \hat{k} \cdot \hat{i} \\
\hat{i} \cdot \hat{j} & \hat{j} \cdot \hat{j} & \hat{k} \cdot \hat{j} \\
\hat{i} \cdot \hat{k} & \hat{j} \cdot \hat{k} & \hat{k} \cdot \hat{k}
\end{bmatrix}
\begin{bmatrix}
\hat{d}_{ix} \\
\hat{d}_{iy} \\
\hat{d}_{iz}
\end{bmatrix}
\]

\[
\begin{align*}
(\hat{i} \cdot \hat{k}) &= (\hat{k} \cdot \hat{i}) = 0 \\
(\hat{j} \cdot \hat{k}) &= (\hat{k} \cdot \hat{j}) = 0 \\
(\hat{k} \cdot \hat{k}) &= 1 \\
(\hat{j} \cdot \hat{i}) &= -\cos\left(\frac{\pi}{2} - \theta\right) = -\cos\left(\frac{\pi}{2}\right)\cos(\theta) - \sin\left(\frac{\pi}{2}\right)\sin(\theta) = -\sin(\theta) \\
(\hat{i} \cdot \hat{j}) &= \cos\left(\frac{\pi}{2} - \theta\right) = +\sin(\theta) \\
(\hat{i} \cdot \hat{i}) &= \cos(\theta) \\
(\hat{j} \cdot \hat{j}) &= \cos(\theta)
\end{align*}
\]
Lecture 11: Frame and Grid Elements.

- (§5.1) the rotation matrix becomes:

\[
\begin{bmatrix}
  d_{ix} \\
  d_{iy} \\
  d_{iz}
\end{bmatrix} =
\begin{bmatrix}
  c\theta & -s\theta & 0 \\
  s\theta & c\theta & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  d_{ix} \\
  d_{iy} \\
  d_{iz}
\end{bmatrix}
\]

Same result as in our investigation of planar rotations but padded to include the third, z, dimension (§3.3)

- The rotation matrix converts representation of any vector from elemental to global coordinates.

- We can use the rotation matrix to replace the elemental representations of the loads and displacements in the element equations.
Lecture 11: Frame and Grid Elements.

The element equations in terms of the global coordinates become:

\[
\begin{bmatrix}
\hat{f}_{1x} \\
\hat{f}_{1y} \\
\hat{m}_{1z} \\
\hat{f}_{2x} \\
\hat{f}_{2y} \\
\hat{m}_{2z}
\end{bmatrix} =
\begin{bmatrix}
c \theta & -s \theta & 0 & 0 & 0 & 0 \\
s \theta & c \theta & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & c \theta & -s \theta & 0 \\
0 & 0 & 0 & s \theta & c \theta & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
f_{1x} \\
f_{1y} \\
m_{1z} \\
f_{2x} \\
f_{2y} \\
m_{2z}
\end{bmatrix}
\]

\[
\{T \hat{f}\} = \hat{k} \{T d\}
\]

\[
f = \begin{bmatrix} T^T \hat{k} T \end{bmatrix} d
\]

\[
k
\]

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Lecture 11: Frame and Grid Elements.

- The final form of the frame element equation is given by Eq. (5.1.11) pg.# 192.

\[
k = \frac{E}{L} \begin{bmatrix}
AC^2 + \frac{12I}{L^2} S^2 & \left( A - \frac{12I}{L^2} \right) CS & -\frac{6I}{L} S & -\left( AC^2 + \frac{12I}{L^2} S^2 \right) & -\left( A - \frac{12I}{L^2} \right) CS & -\frac{6I}{L} S \\
AS^2 + \frac{12I}{L^2} C^2 & \frac{6I}{L} C & -\left( A - \frac{12I}{L^2} \right) CS & -\left( AS^2 + \frac{12I}{L^2} C^2 \right) & \frac{6I}{L} C \\
4I & \frac{6I}{L} S & -\frac{6I}{L} C & 2I \\
AC^2 + \frac{12I}{L^2} S^2 & \left( A - \frac{12I}{L^2} \right) CS & \frac{6I}{L} S \\
AS^2 + \frac{12I}{L^2} C^2 & -\left( A - \frac{12I}{L^2} \right) CS & -\frac{6I}{L} C & 4I
\end{bmatrix} \text{SYM}
\]

\[
k = \frac{E}{L} \begin{bmatrix}
AC^2 + \frac{12I}{L^2} S^2 & \left( A - \frac{12I}{L^2} \right) CS & -\frac{6I}{L} S & -\left( AC^2 + \frac{12I}{L^2} S^2 \right) & -\left( A - \frac{12I}{L^2} \right) CS & -\frac{6I}{L} S \\
AS^2 + \frac{12I}{L^2} C^2 & \frac{6I}{L} C & -\left( A - \frac{12I}{L^2} \right) CS & -\left( AS^2 + \frac{12I}{L^2} C^2 \right) & \frac{6I}{L} C \\
4I & \frac{6I}{L} S & -\frac{6I}{L} C & 2I \\
AC^2 + \frac{12I}{L^2} S^2 & \left( A - \frac{12I}{L^2} \right) CS & \frac{6I}{L} S \\
AS^2 + \frac{12I}{L^2} C^2 & -\left( A - \frac{12I}{L^2} \right) CS & -\frac{6I}{L} C & 4I
\end{bmatrix} \text{SYM}
\]

\[
C = \cos(\theta) = c\theta \quad S = \sin(\theta) = s\theta
\]