
- We will focus initially on the steady state heat transfer problem.
- Start by looking at the transfer of thermal energy along one dimension.
  - With no convection off of the perimeter surface (insulated).
  - With convection off of the perimeter surface.
- Our analysis will be based on a discretization of the governing DEs using Galerkin’s method.
  - §13.4 of Logan applies a variational formulation of the heat transfer element equations.
  - §13.8 switches to Galerkin’s method due to the addition of mass transport terms that complicate the variational method.
- We still use Steps 1 and 2 as illustrated in §13.4. If you read the rest of the §13.4 you will see why we are using a weighted residual technique.


- First problem addressed is 1-D Heat Conduction with no convection.
- The finite element is a region in space.
  - The boundaries of the region are defined by fixed points (or nodes).
  - There are quantities of interest at the boundaries of the region – temperatures and heat flux values.
  - Want to calculate the values for this discrete set of entities (the state variables).
  - For 1-D problems this region can be considered as a filament (a line element).
  - The fluctuations of the state variables along the line are assumed to be representative of what occurs over the whole region.
Fourier’s law of heat conduction:

\[ q_x = -K_{xx} \frac{dT}{dx} \]

- \( q_x \) \equiv heat conducted (heat flux) kW/m²
- \( K_{xx} \) \equiv thermal conductivity in x direction kW/(°C · m)
- \( \frac{dT}{dx} \) \equiv temperature gradient °C/m
- \( c \equiv \) specific heat capacity kWh/(kg · °C)
- \( E \equiv \) energy kWh
- \( U \equiv \) total internal thermal energy kWh
- \( Q \equiv \) rate of internal heat generation kW/m³

- The 1st law of thermodynamics states that there is a balance of the energy produced, stored, and conducted through the finite element (control volume):

\[ E_{in} + Q_{TOT} = \Delta U + E_{out} \]

\[ q_x A dt + Q A dx dt = \Delta U + q_{x+dx} A dt \]

\[ q_x = -K_{xx} \frac{dT}{dx} \]

\[ q_{x+dx} = -\left[ K_{xx} \frac{dT}{dx} + \frac{d}{dx}\left(K_{xx} \frac{dT}{dx}\right) dx \right] \]

\[ \Delta U = c \left( \rho A dx \right) dT \]

- Substituting these expressions into the energy balance…

\[
\frac{d}{dx} \left( K_{xx} \frac{dT}{dx} \right) + Q = \rho c \frac{dT}{dt}
\]

- If the temperature profile can be assumed to be steady:

\[
\frac{d}{dx} \left( K_{xx} \frac{dT}{dx} \right) + Q = 0
\]
§13.3. Typical Units; thermal Conductivities, $K$; and Heat Transfer Coefficients, $h$.

- Tables 13-1 through 13-3 list common $K$ and $h$ values and define the units of various thermal quantities.
- We will visit the use of the convective coefficients when we visit §13.2 in Lecture 23.
- The convective coefficients listed are for free convection.
- In forced convection there is a velocity of the liquid/gas created by an external source which would increase the effective $h$ value.

- The first law of thermodynamics has produced a differential equation that defines a temperature distribution.
- The solution is a function $T(x)$: the temperature distribution along the dimension $x$.
- The initial stage of the problem statement reduced us to a 1-D DE. We assumed there was heat transfer in only the $x$ direction.
- Calculation of the true solution to the differential equation will involve applying boundary conditions to solve for constants of integration.
- We will look at the FE approximation to the true solution.
- We expect to have boundary terms exposed in our formulation of the element equations.
Consider what the boundary terms might be...

Boundary Conditions (case study):
- At the left edge of the domain:
  \[ T = T_B \]
- At the right edge of the domain:
  \[ q_x = -K_{xx} \frac{dT}{dx} = q_x^* \]

Insulated boundaries ensuring a 1-D heat conduction.
P. 13.4 (modified), pg.# 500.

The rod of 1 in. radius shown below generates heat internally at the rate of $Q=10,000$ Btu/(hft$^3$) throughout the rod. The left edge and perimeter of the rod are insulated, and the right edge is exposed to an ambient temperature of 100 deg F. The thermal conductivity of the rod is $K_{xx}=12$ Btu/(h-ft-degF) and the length of the rod is $L=3$ in. Calculate the temperature distribution along the rod axis assuming that there is no convection over the perimeter of the rod and that the convection off of the right end is ideal. Use at least three elements.

$$T_B = T_\infty = 100 \degree F$$
Governing differential equation:

\[ \frac{d}{dx}\left( K_{xx} \frac{dT}{dx} \right) + Q = 0 \]

Fourier’s law defines the heat flux:

\[ q_x = -K_{xx} \frac{dT}{dx} \]

Provided an approximate temperature distribution we can recover the heat flux.

A conforming element requires at least a 1st order approximation to the temperature distribution.
Consider the boundary conditions:

- At the left end of the rod (flux condition):
  \[ q_x \big|_{x=0} = \left( -K_{xx} \frac{dT}{dx} \right)_{x=0} = 0 \]

- At the right end of the rod:
  \[ T \big|_{x=3.0} = 100 \]
§13.4 One dimensional Finite Element Formulation (using a WR method).

- We can use the weighted residuals method to get element equations that define a linear segment of an approximate temperature profile.

\[ T(x) = T_1 + \frac{T_2 - T_1}{x_2 - x_1}(x - x_1) \]

- \( x = 0 \)
- \( x = x_1 \)
- \( x = x_2 \)
- \( x = 3 \)

- \( T_B = T_\infty = 100 \degree F \)

This problem is very similar to that completed in Lecture #16.

\[
\hat{U} = NU
\]

\[
N = \begin{bmatrix} \left(1 - \frac{\hat{x}}{L}\right) & \left(\frac{\hat{x}}{L}\right) \end{bmatrix}_{1 \times 2}
\]

\[
U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}
\]

\[
\hat{x} = x - x_1
\]

\[
x = \hat{x} + x_1
\]

\[
\therefore \frac{d\hat{x}}{dx} = 1
\]