Chapter #16: Structural Dynamics and Time Dependent Heat Transfer.

Lectures #1-26 have discussed only ‘steady’ systems.

There has been no time dependence in any problems.

We will investigate beam dynamics and show the additional steps in the numerical solution of a time dependent problem.

There are many time dependent problems discussed in Chapter #16.

§16.1 – 1 DOF spring mass system.

§16.2, 16.4 & 16.5 – bar element dynamic response.

§ 16.7 – truss and plane frame analysis.

§ 16.8 – time dependent (unsteady) heat transfer.
§16.6 Beam Element Mass Matrices and Natural Frequencies.

- We will revisit the formation of the governing DE for the equilibrium of the beam element.
- Equilibrium now involves an ‘inertial force’ acting on the differential beam element.
- The approximate displacement field $v(x)$ will be revised to include the time dependence.
- We will see the generation of the consistent mass matrix through the evaluation of the weighted residuals.
- We will look at the assembly process for a beam dynamics problem and discuss the computational advantages of a ‘lumped mass approximation.’
- The solution process will be expanded to include the integration of the state vector ($§16.3$).
- Define the phrase *initial conditions*. 
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- Beam Dynamics:

The differential element is now accelerating in the positive $y$ direction in reaction to the external forces.
Consider what happens as a beam element moves (vibrates or translates in space).

- The profile of our element is defined by node coordinates and node rotations.
- The nodal values (the state vector $d$) is blended by the shape function matrix.
- For the moving beam the profile is fluctuating.
- In the case shown the slopes at both nodes 1 and 2 and the vertical displacement at node 1 have changed over an interval $\Delta t$. 
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- The approximation must be time dependent.

\[ \hat{v}(\hat{x}) = N \cdot \hat{d} \]

\[
= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} \hat{d}_{1y} & \phi_1 & \hat{d}_{2y} & \phi_2 \end{bmatrix}^T
\]

The shape functions are dependent on space. They are set to satisfy element boundary conditions.

\[
N_1 = \frac{1}{L^3} \left( 2\hat{x}^3 - 3\hat{x}^2 L + L^3 \right)
\]

\[
N_2 = \frac{1}{L^3} \left( \hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x} L^3 \right)
\]

\[
N_3 = \frac{1}{L^3} \left( -2\hat{x}^3 + 3\hat{x}^2 L \right)
\]

\[
N_4 = \frac{1}{L^3} \left( \hat{x}^3 L - \hat{x}^2 L^2 \right)
\]

So, the node state vector becomes the time dependent quantity.

\[
\hat{v}(\hat{x}, t) = N(\hat{x}) \cdot \hat{d}(t)
\]

\[
\dot{\hat{v}}(\hat{x}) = N(\hat{x}) \cdot \dot{\hat{d}}(t)
\]

\[
\ddot{\hat{v}}(\hat{x}) = N(\hat{x}) \cdot \ddot{\hat{d}}(t)
\]
A force balance on the differential element gives:

\[-dV - w(\hat{x})d\hat{x} = (\rho A d\hat{x}) \ddot{v}(\hat{x})\]

We can evaluate our approximate inertial term.

\[
\hat{V} = EI \frac{d^3 \hat{v}(\hat{x})}{d\hat{x}^3}
\]

\[
EI \frac{\partial^4 \hat{v}(\hat{x}, t)}{\partial \hat{x}^4} + w + (\rho A) \frac{\partial^2 \hat{v}(\hat{x}, t)}{\partial^2 t} = 0
\]

A new term in our residual equations for the beam element. Note that the shape functions have not changed.
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- Since the shape functions have not changed, the contributions of the first two terms to the element equations will not change.
- We need only evaluate the residual term produced by the acceleration term in the motion equation.
- We do this in a consistent manner: we try to drive four weighted residuals to zero using Galerkin’s choice of weighting functions.
- This leads to the phrase ‘consistent mass matrix’.

\[
\int_{\hat{x}=0}^{\hat{x}=L} N^T \left\{ EI \frac{\partial^4 \hat{v}(\hat{x}, t)}{\partial \hat{x}^4} \right\} \cdot d\hat{x} + \int_{\hat{x}=0}^{\hat{x}=L} N^T \left\{ w \right\} \cdot d\hat{x} + \int_{\hat{x}=0}^{\hat{x}=L} N^T \left\{ (\rho A) \frac{\partial^2 \hat{v}(\hat{x}, t)}{\partial^2 t} \right\} \cdot d\hat{x} = 0
\]
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- Following the MWR procedure:

\[
I_a = \int_{\hat{x}=0}^{\hat{x}=L} \left( \rho A \frac{\partial^2 \hat{v}(\hat{x}, t)}{\partial^2 t} \right) \cdot d\hat{x}
\]

\[
I_a = (\rho A) \int_{\hat{x}=0}^{\hat{x}=L} \left( \begin{bmatrix} N \end{bmatrix} \ddot{d} \right) \cdot d\hat{x}
\]

\[
I_a = \left[ (\rho A) \int_{\hat{x}=0}^{\hat{x}=L} N^T N \cdot d\hat{x} \right] \ddot{d} \rightarrow \ddot{d} = \begin{bmatrix} \ddot{d}_{1y} & \ddot{\phi}_1 & \ddot{d}_{2y} & \ddot{\phi}_2 \end{bmatrix}^T
\]

The consistent mass matrix.
When we append this inertial term onto the results of our previous weighted residual evaluations:

\[
\begin{bmatrix}
\hat{f}_{1y}
m_{1z}
\hat{f}_{2y}
m_{2z}
\end{bmatrix} + \begin{bmatrix}
\hat{m}_{1z}
m_{1z}
\hat{m}_{2z}
m_{2z}
\end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix}
12 & 6L & -12 & 6L \\
4L^2 & -6L & 2L^2 & \\
12 & -6L & 4L^2 & \\
\end{bmatrix} \begin{bmatrix}
\hat{d}_{1y}
\phi_1 \\
\hat{d}_{2y}
\phi_2 \\
\end{bmatrix} + \frac{\rho AL}{420} \begin{bmatrix}
156 & 22L & 54 & -13L \\
4L^2 & 13L & -3L^2 & \\
156 & -22L & 4L^2 & \\
\end{bmatrix} \begin{bmatrix}
\ddot{d}_{1y} \\
\ddot{\phi}_1 \\
\ddot{d}_{2y} \\
\ddot{\phi}_2 \\
\end{bmatrix}
\]

Our approximate displacement field keeps the element equations linear (we do not see products of state variables).

In MECH 330 you learned how to acquire the natural frequencies and mode shapes for a multi-DOF undamped system.
An alternative to the consistent mass matrix is the lumped mass matrix.

The lumped mass representation is a heuristic discretization of mass.

The consistent mass matrix is formed by using the actual mass distribution within the residual evaluations.

The lumped approximation assumes that the beam is very slender and it lumps the mass at node points.

There is no inertia associated with changing node slopes.

The lumped approximation is used because it diagonalizes the mass matrix.

Lumping is not a mathematical process – it is an idealization that is executed on the element equations.
Half of the beam’s total mass is at node 1.

Half of the beam’s total mass is at node 2.

The fluctuations of the curvilinear profile do not accelerate any mass.
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- When we apply the lumped mass approximation:

\[
\begin{bmatrix}
\hat{f}_{1y} \\
\hat{m}_{1z} \\
\hat{f}_{2y} \\
\hat{m}_{2z}
\end{bmatrix} + \begin{bmatrix}
\hat{f}_{1y}^w \\
\hat{m}_{1z}^w \\
\hat{f}_{2y}^w \\
\hat{m}_{2z}^w
\end{bmatrix} = \begin{bmatrix}
\frac{EI}{L^3} & 0 & \frac{EIL}{L^3} \\
0 & \frac{4EI}{L^3} & -\frac{2EI}{L^3} \\
\frac{EIL}{L^3} & -\frac{2EI}{L^3} & \frac{2EI}{L^3}
\end{bmatrix} \begin{bmatrix}
\hat{\phi}_1 \\
\hat{\phi}_2
\end{bmatrix} + \rho A L \begin{bmatrix}
1 \\
0
\end{bmatrix} \begin{bmatrix}
\ddot{d}_{1y} \\
\ddot{d}_{2y}
\end{bmatrix} + \begin{bmatrix}
1 \\
0
\end{bmatrix} \begin{bmatrix}
\dddot{d}_{1y} \\
\dddot{d}_{2y}
\end{bmatrix}
\]

- We will see that the lumped mass approximation eases the computational burden of modal analysis and time domain simulation (Lecture #28).
We know that in the undamped free vibration case, the transverse vibration at any point along the element will be harmonic.

Consider free (natural) vibration.

\[ 0 = \hat{k} \cdot \hat{d} + \hat{m} \cdot \ddot{\hat{d}} = (\hat{k} \cdot D - \omega_n^2 \hat{m} \cdot D) e^{i\omega_n t} = (\hat{k} - \omega_n^2 \hat{m}) \cdot D e^{i\omega_n t} \]

Magnitudes of the 4 vibration signals.

Must be a singular matrix.

An assumed harmonic variation of the state variables.

The solution to the undamped free vibration problem is the same as the eigen problem for the system dynamic matrix.
The $\omega^2$ values are the eigenvalues of the dynamic matrix.

The $D$ vectors are the eigenvectors of the dynamic matrix.
Example problem 16.5:

We now consider the determination of the natural frequencies of vibration for a beam fixed at both ends as shown in Figure 16–16. The beam has mass density $\rho$, modulus of elasticity $E$, cross-sectional area $A$, area moment of inertia $I$, and length $2L$. For simplicity of the longhand calculations, the beam is discretized into two elements of length $L$.

![Figure 16–16  Beam for determination of natural frequencies](image)
Lecture 27: Structural Dynamics - Beams.

- Forming the consistent element equations.

\[
\begin{align*}
\begin{bmatrix} f_{1y}^{(1)} \\ m_{1z}^{(1)} \\ f_{2y}^{(1)} \\ m_{2z}^{(1)} \end{bmatrix} &= \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 4L^2 & -6L & 2L^2 \\ 12 & -6L & 4L^2 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{bmatrix} + \frac{\rho AL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 4L^2 & 13L & -3L^2 \\ 156 & -22L \\ 4L^2 \end{bmatrix} \begin{bmatrix} \ddot{d}_{1y} \\ \ddot{\phi}_1 \\ \ddot{d}_{2y} \\ \ddot{\phi}_2 \end{bmatrix}
\end{align*}
\]

Element #1

\[
\begin{align*}
\begin{bmatrix} f_{2y}^{(2)} \\ m_{2z}^{(2)} \\ f_{3y}^{(2)} \\ m_{3z}^{(2)} \end{bmatrix} &= \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 4L^2 & -6L & 2L^2 \\ 12 & -6L & 4L^2 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{bmatrix} + \frac{\rho AL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 4L^2 & 13L & -3L^2 \\ 156 & -22L \\ 4L^2 \end{bmatrix} \begin{bmatrix} \ddot{d}_{2y} \\ \ddot{\phi}_2 \\ \ddot{d}_{3y} \\ \ddot{\phi}_3 \end{bmatrix}
\end{align*}
\]

Element #2
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- Assembling and applying the homogeneous boundary conditions.

\[
\begin{bmatrix}
  f_{1y}^{(1)} \\
  m_{1z}^{(1)} \\
  f_{2y}^{(1)} + f_{2y}^{(2)} \\
  m_{2z}^{(1)} + m_{2z}^{(2)} \\
  f_{3y}^{(2)} \\
  m_{3z}^{(2)} \\
  0
\end{bmatrix} = \begin{bmatrix}
  f_{1y} \\
  m_{1z} \\
  f_{2y} \\
  m_{2z} \\
  f_{3y} \\
  m_{3z} \\
  0
\end{bmatrix} = \begin{bmatrix}
  12 & 6L & -12 & 6L & 0 & 0 \\
  6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\
  -12 & -6L & 12 + 12 & -6L + 6L & -12 & 6L \\
  6L & 2L^2 & -6L + 6L & 4L^2 + 4L^2 & -6L & 2L^2 \\
  0 & 0 & -12 & -6L & 12 & -6L \\
  0 & 0 & 6L & 2L^2 & -6L & 4L^2 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
  d_{1y} \\
  d_{2y} \\
  \phi \\
  d_{3y} \\
  \phi_2 \\
  \phi_3
\end{bmatrix}
\]

\[
+ \frac{\rho AL}{420} \begin{bmatrix}
  156 & 22L & 54 & -13L & 0 & 0 \\
  22L & 4L^2 & 13L & -3L^2 & 0 & 0 \\
  54 & 13L & 156 + 156 & -22L + 22L & 54 & -13L \\
  -13L & -3L^2 & -22L + 22L & 4L^2 + 4L^2 & 13L & -3L^2 \\
  0 & 0 & 54 & 13L & 156 & -22L \\
  0 & 0 & -13L & -3L^2 & -22L & 4L^2
\end{bmatrix} \begin{bmatrix}
  \ddot{d}_{1y} \\
  \ddot{d}_{2y} \\
  \ddot{\phi} \\
  \ddot{d}_{3y} \\
  \ddot{\phi}_2 \\
  \ddot{\phi}_3
\end{bmatrix}
\]
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- The application of the BC’s is what specifies the system nature.
- The natural frequencies we calculate are now specific to the fixed-fixed supports.

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \begin{bmatrix} d_{2y} \\ \phi_2 \end{bmatrix} + \frac{\rho AL}{420} \begin{bmatrix} 312 & 0 \\ 0 & 8L^2 \end{bmatrix} \begin{bmatrix} \ddot{d}_{2y} \\ \ddot{\phi_2} \end{bmatrix}
\]

- The associated eigen problem is:

\[
\begin{bmatrix} EI & 0 \\ 0 & \frac{420}{8L^2} \end{bmatrix} - \omega^2 \begin{bmatrix} \rho AL & 0 \\ 0 & 8L^2 \end{bmatrix} = 0
\]

- The natural frequencies (x2) are given by:

\[
\omega_1^2 = \frac{420EI}{13\rho AL^4} \quad \therefore \omega_1 = \frac{5.68}{L^2} \sqrt[2]{\frac{EI}{\rho A}} \quad ; \quad \omega_2^2 = \frac{420EI}{\rho AL^4} \quad \therefore \omega_2 = \frac{20.49}{L^2} \sqrt[2]{\frac{EI}{\rho A}}
\]