5 Pure Bending of Beams

5-1. INTRODUCTION

The system of forces that may exist at a section of a beam was discussed in the previous chapter. This was found to consist of an axial force, a shearing force, and a bending moment. The effect of one of these forces, the axial force, on a member was discussed in Chapters 1 and 2. In this chapter another element of the force system that may be present at a section of a member, the internal bending moment, will be considered. Moreover, since in some cases a segment of a beam may be in equilibrium under the action of a moment alone, a condition called pure bending or flexure, this in itself represents a complete problem. It is the purpose of this chapter to relate the internal bending moment to the stresses it causes in a beam. If, in addition to the internal bending moment, an axial force and a shear also act simultaneously, complex stresses arise. These will be treated in Chapters 7, 8, and 9. The deflection of beams due to bending will be discussed in Chapter 11.

A major part of this chapter will be devoted to methods for determining the stresses in straight homogeneous beams caused by bending moments. Topics on beams made from two or more different materials, curved beams, and stress concentrations are also included.

5-2. SOME IMPORTANT LIMITATIONS OF THE THEORY

Just as in the case of axially loaded rods and in the torsion problem, all forces applied to a beam will be assumed to be steady and delivered to the beam without shock or impact. Shock or impact problems will be considered in Chapter 15. Moreover, all of the beams will be assumed to be stable under the applied forces. A similar point was brought out in Chapter 1, where it was indicated that a rod acting in compression cannot be too slender, or its behavior will not be governed by the usual compressive strength criterion. In such cases the stability of the member becomes important. As an example, consider the possibility of using a sheet of paper on edge as a beam. Such a beam has a substantial depth, but even if it is used
to carry a force over a small span, it will buckle sideways and collapse. The same phenomenon may take place in more substantial members which may likewise collapse under an applied force. Such unstable beams do not come within the scope of this chapter. All the beams considered here will be assumed to be sufficiently stable laterally by virtue of their proportions, or to be thoroughly braced in the transverse direction. A better understanding of this important phenomenon will result after the study of the chapter on columns. The majority of beams used in structural framing and machine parts are such that the flexural theory to be developed here is applicable. This is indeed fortunate as the theory governing the stability of members is more complex.

5-3. BASIC ASSUMPTIONS

For the present it is assumed that only straight beams having constant cross-sectional areas with an axis of symmetry are to be included in the discussion. Moreover, it is assumed that the applied bending moments lie in a plane containing this axis of symmetry and the beam axis. Let it be further agreed that for the sake of simplicity in making sketches, the axis of symmetry will be taken vertically. Several cross-sectional areas of beams satisfying these conditions are shown in Fig. 5-1. A generalization of this problem will be made in Art. 5-7.

A segment of a beam fulfilling the above requirements is shown in Fig. 5-2(a), and its cross-section is shown in Fig. 5-2(b). For such a beam a line through the centroid of all cross-sections will be referred to as the axis of the beam. Next, imagine that two planes are passed through the beam perpendicular to its axis. The intersections of these planes with a longitudinal plane passing through the beam axis and the axis of symmetry is shown by lines \( AB \) and \( CD \). Then it is not difficult to imagine that when this segment is subjected to the bending moments \( M \) at its ends as shown in Fig. 5-2(c), the beam bends, and the planes perpendicular to the beam axis tilt slightly. Moreover, the lines \( AB \) and \( CD \) remain straight.* This can be satisfactorily verified experimentally.† Generalizing this observation for the whole beam,

*This can be demonstrated by using a rubber model with a ruled grating drawn on it. Alternatively, thin vertical rods passing through the rubber block can be used. In the immediate vicinity of the applied moments the deformation is more complex. However, in accord with the St. Venant’s principle (Art. 2-11), this is only a local phenomenon which rapidly dissipates.

†Rigorous solutions from the Mathematical Theory of Elasticity show that slight warpage of these lines may take place. Such warpage occurs if a beam carries a shear in addition to a bending moment. However, the warpage of the adjoining sections is exceedingly similar in shape. Thus the distance between any two points such as \( A \) and \( C \) on the adjoining sections remains practically the same whether warped or straight lines \( AB \) and \( CD \) are considered. And since the distance between the adjoining sections is the basis for establishing the elementary flexure theory, the foregoing assumption forms an excellent working hypothesis for all cases. Moreover, a conclusion of far-reaching importance is that the existence of a shear at a section does not invalidate the expressions to be derived in this chapter. This will be implied in the subsequent work.
ays and collapse. The members which may
beams do not come
considered here will be their proportions, or
better understanding of the chapter on
framing and machine
ed here is applicable.

Fig. 5-1. Beam cross sections with a vertical axis of symmetry

(a) Beam axis

(b) Centroid

(c) +M

(d) Initial length

(e) Unit length

Fig. 5-2. Behavior of a beam in bending

ART. 5-3 BASIC ASSUMPTIONS
one obtains the most fundamental hypothesis of the flexure theory, based on the geometry of deformations. It may be stated thus:

1. Plane sections through a beam, taken normal to its axis, remain plane after the beam is subjected to bending.

This means that in a bent beam two planes normal to the beam axis and initially parallel cease to be parallel. In a side view, the behavior of two such planes corresponds to the behavior of lines $AB$ and $CD$ of Figs. 5-2(a) and (c). An element of the beam contained between these planes is shown in Fig. 5-2(d). Under the action of the moments of the sense shown, the distance $AC$ becomes smaller than $BD$. Further, because of the internal moment, a push must exist on the upper part of the beam and a pull on the lower. Hence, the undistorted beam element must be related to the distorted one, as $A'C'D'B'$ is to $ACDB$, shown in more detail in Fig. 5-2(d). From this diagram it is seen that the fibers or "filaments" of the beam along the surface $ab$ do not change in length. Hence, the fibers in the surface $ab$ are not stressed at all, and, as the element selected was an arbitrary one, fibers free of stress exist continuously over the whole length and width of the beam. These fibers lie in a surface which is called the neutral surface of the beam. Its intersection with a right section through the beam is termed the neutral axis of the beam. Either term implies a location of zero stress in the member subjected to bending.

The precise location of the neutral surface in a beam will be determined in the next article. First, a study of the nature of the strains in fibers parallel to the neutral surface will be made. Thus, consider a typical fiber such as $cd$ parallel to the neutral surface and located at a distance $y$ from it. During bending it elongates an amount $\Delta$. If this elongation is divided by the initial length $L$ of the fiber, the strain $\varepsilon$ in that fiber is obtained. Next, note that from the geometrical assumption made earlier, elongations of different fibers vary linearly from the neutral axis since these elongations are fixed by the triangles $aBB'$, $bDD'$, $aAA'$, and $bCC'$. On the other hand, the initial length of all fibers is the same. Hence the original fundamental assumption may be restated thus:

1a. In a beam subjected to bending, strains in its fibers vary linearly or directly as their respective distances from the neutral surface.

This situation is analogous to the one found earlier in the torsion problem where the shearing strains vary linearly from the axis of a circular shaft. In a beam, strains vary linearly from the neutral surface. This variation is represented diagrammatically in Fig. 5-2(e). These axial strains are

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*This hypothesis with an inaccuracy was first introduced by Jacob Bernoulli (1645-1705), a Swiss mathematician. In the correct form it dates back to the writings of the French engineering educator M. Navier (1785-1836).

†A rigorous solution shows that this surface is slightly cylindrical in two directions. In the present treatment this surface is assumed to be curved only in the direction shown.

‡Positive direction of $y$ is taken upward from the neutral axis.

§Experimentally, this assumption may be more easily verified than assumption (1).
associated with stresses which act normal to the section of a beam. The above corollary to the original assumption is applicable in the elastic as well as in the inelastic range of the material’s behavior. For the present this generality will be limited by introducing the second fundamental assumption of the flexure theory:

2. Hooke’s law is applicable to the individual fibers, i.e., stress is proportional to strain. The same elastic modulus $E$ is assumed to apply to material in tension as well as in compression. The Poisson effect and the interference of the adjoining differently stressed fibers are ignored.

Combining the foregoing assumptions, the basis for establishing the flexural theory for the elastic case is obtained:

On a section of a beam, normal stresses resulting from bending vary linearly as their respective distances from the neutral axis.

It should be firmly fixed in the reader’s mind that these stresses act normal to the section of a beam. They are the result of axial elongation or contraction of the various beam fibers. Their linear variation from the neutral axis, to repeat, is due to the linear variation of the strains and to the proportionality of stress to strain. The distance to the various fibers of the beam is measured vertically from the neutral axis. Figures 5-3(a) and (b) illustrate the nature of the stress distribution in a beam resisting a bending moment. Two alternative schemes of representing this three-dimensional problem in a plane are shown in Figs. 5-3(c) and (d). In subsequent work these will be the usual forms for showing the flexural stress distribution at a section of a beam.

5-4. THE FLEXURE FORMULA

After the nature of the stress distribution in the elastic range at a section of a beam is understood, quantitative expressions relating bending moment to stress may be established. For this purpose the neutral surface is first located from considerations of static equilibrium.

Consider a beam segment subjected to a positive bending moment $M$ as shown in Fig. 5-4(a). At section $X-X$ this applied moment is resisted by stresses which vary linearly from the neutral axis. The highest stresses occur at the points most remote from the neutral axis. For the beam shown this occurs along the line ed, Fig. 5-4(b). This stress, being a normal stress, is designated by $\sigma_{max}$, Fig. 5-4(a). Any other normal stress acting on the cross-sectional area is related to this stress by a ratio of distances from the neutral axis.
axis. Thus, on an infinitesimal area $dA$, Fig. 5-4(b), at a distance $y$ from the neutral axis, the stress is $-(y/c) \sigma_{\text{max}}$, where distance $c$ is measured from the neutral axis to the most remote fiber of the beam. Stresses below the neutral surface are given by a similar relation; the sign automatically reverses as $y$'s are measured down from the neutral axis. This reversal of sign corresponds to the reversal in stress from compression to tension. Note that for a positive bending moment, the normal stresses at a section are positive (tension) for negative values of $y$, and negative (compression) for positive values of $y$. Hence the expression $-(y/c) \sigma_{\text{max}}$ is a general expression for the normal stress on any infinitesimal area of the beam's section at a distance $y$ from the neutral axis.

Since the segment of the beam shown in Fig. 5-4(a) must be in equilibrium, the sum of all forces in the $x$-direction, which is taken horizontally, must vanish, i.e., $\sum F_x = 0$. Therefore as the beam’s segment resists only a couple, the sum (or integral) of all forces acting on the section of the beam must vanish. Thus

$$\int_A \left( -\frac{y}{c} \sigma_{\text{max}} \right) \frac{dA}{(\text{stress})(\text{area})} = 0$$

where the subscript $A$ of the integral indicates that the summation must be carried out over the entire cross-sectional area of the beam. At a section, however, $\sigma_{\text{max}}$ and $c$ are constants, so the integral may be rewritten as

$$-\frac{\sigma_{\text{max}}}{c} \int_A y \, dA = 0$$

Since in a stressed beam neither $c$ nor $\sigma_{\text{max}}$ can be zero, it follows that $\int_A y \, dA = 0$. But by definition $\int_A y \, dA = \bar{y} A$, where $\bar{y}$ is the distance from a baseline (neutral axis in the case considered) to the centroid of the area $A$, so $\bar{y} A = 0$. Then since $A$ is not zero, $\bar{y}$ must be. Therefore the distance from the
neutral axis to the centroid of the area must be zero, and the neutral axis passes through the centroid of the cross-sectional area of the beam. Hence the neutral axis may be quickly and easily determined for any beam by simply finding the centroid of the cross-sectional area.

Next, the remaining significant equation of static equilibrium will be applied to the beam segment shown in Fig. 5-4(a) to evaluate the magnitudes of the normal stresses. This equation is \( \sum M = 0 \), which for the present purpose is more conveniently stated as: The external moment \( M \) is resisted by or equal to the internal bending moment developed by the flexural stresses at a section. The latter quantity is determined by summing forces acting on infinitesimal areas \( dA \), Fig. 5-4(b), multiplied by their respective arms from the neutral axis. By formulating these statements mathematically, the following equality is obtained:

\[
M = \int_A \left( \sigma_{\text{max}} \right) \frac{y}{c} \, dA \\
\text{where (as before) } \sigma_{\text{max}}/c \text{ is a constant, hence it appears outside the integral}
\]

The integral \( \int_A y^2 \, dA \) depends only on the geometric properties of the cross-sectional area. In mechanics of materials this integral is called the moment of inertia of the cross-sectional area about the centroidal axis, when \( y \) is measured from such an axis. It is a definite constant for any particular area, and in this text it will be designated by \( I \). With this notation the foregoing expression may be written more compactly as

\[
M = -\frac{\sigma_{\text{max}}}{c} I \quad \text{or} \quad \sigma_{\text{max}} = -\frac{Mc}{I}
\]

It is customary to dispense with the sign for the normal stress as its sense can be found by inspection. At any section the normal stresses must act in such a manner as to build up a couple statically equivalent to the resisting bending moment, the sense of which is known. Hence the above equation can be written simply as

\[
\sigma_{\text{max}} = \frac{Mc}{I}
\]

Equation 5-1 is the flexure formula* for beams. It gives the maximum

*It took nearly two centuries to develop this seemingly simple expression. The first attempts to solve the flexure problem were made by Galileo in the seventeenth century. In the form in which it is used today the problem was solved in the early part of the nineteenth century. Generally, Navier of France is credited for this accomplishment. However, some maintain that credit should go to Coulomb, who also derived the torsion formula.
normal stress in a beam subjected to a bending moment $M$. Moreover, since stress $\sigma$ on any point of a cross-section is $-(y/c)\sigma_{max}$, the general expression for normal stresses at a section is given as

$$\sigma = -\frac{My}{I}$$  \hspace{1cm} (5-1a)

These formulas are of unusually great importance in mechanics of materials. In these formulas, $M$ is the internal or resisting bending moment, which is equal to the external moment at the section where the stresses are sought. The bending moment is expressed in newton-meter (N·m) units for use in these formulas. The distance $y$ from the neutral axis of the beam to the point on a section where the normal stress $\sigma$ is wanted is measured perpendicular to the neutral axis and should be expressed in meters. When it reaches its maximum value (measured either up or down) it corresponds to $c$, and as $y$ approaches this maximum value, the normal stress $\sigma$ approaches $\sigma_{max}$. In this equation $I$ is the moment of inertia of the whole cross-sectional area of the beam about its neutral axis. To avoid confusion with the polar moment of inertia, $I$ is sometimes referred to as the rectangular moment of inertia. It has the dimensions of m$^4$. Its evaluation for various areas will be discussed in the next article. The use of consistent units as indicated above makes the units of stress $\sigma$, [N·m][m]/[m$^4$] $= N/m^2$ = Pa.

The student is urged to reflect on the meanings of the terms used in the derived equations. The stresses given by these equations indicate that they act perpendicular to the section and vary linearly from the neutral axis. These facts are very significant. Likewise, the three-dimensional aspect of the problem must be kept in mind.

The foregoing discussion applies only to cases where the material behaves elastically. The important concepts used in deriving the flexure formula may be summarized as follows:

1. **Deformation** was assumed giving the linear variation of strain from the neutral axis.
2. **Properties of materials** were used to relate strain and stress.
3. **Equilibrium conditions** were used to locate the neutral axis and to determine the internal stresses.

These are the same concepts as were used to derive the torsion formula.

**5-5. COMPUTATION OF THE MOMENT OF INERTIA**

In applying the flexure formula, the moment of inertia $I$ of the cross-sectional area about the neutral axis must be determined. Its value is defined by the integral of $y^2 \, dA$ over the entire cross-sectional area of a member, and it must be emphasized that for the flexure formula the moment of
Moreover, since the general expression

\[ (5-1a) \]

in mechanics of bending moment, where the stresses are \( \sigma \) (N/m) units for the neutral axis of the beam to the measured perpendiculars. When it reaches the moment of inertia, areas will be discussed above makes the terms of the equation used in the equations indicate that early from the neutral axis in deriving the flexure of strain from the neutral stress.

\[ \text{INERTIA} \]

of inertia \( I \) of the cross-section. Its value is the definition of a member, formula the moment of inertia must be computed around the neutral axis of the cross-sectional area. This axis, according to Art. 5-4, passes through the centroid of the cross-sectional area. For symmetrical sections the neutral axis is perpendicular to the axis of symmetry. Such an axis is one of the principal axes* of the cross-sectional area. Most readers should already be familiar with the method of determining the moment of inertia \( I \). However, the necessary procedure is reviewed below.

The first step in evaluating \( I \) for an area is to find the centroid of the area. An integration of \( y^2 \, dA \) is then performed with respect to the horizontal axis passing through the area’s centroid. Actual integration over areas is necessary for only a few elementary shapes, such as rectangles, triangles, etc. After this is done, most cross-sectional areas used in practice may be broken down into a combination of these simple shapes. Values of moments of inertia for some simple shapes may be found in any standard civil or mechanical engineering handbook (also see Table 2 of the Appendix). To find \( I \) for an area composed of several simple shapes, the parallel-axis theorem (sometimes called the transfer formula) is necessary; the development of it follows.

The area shown in Fig. 5-5 has a moment of inertia \( I_o \) around the horizontal axis passing through its own centroid, i.e.,

\[ I_o = \int y^2 \, dA, \]

where \( y \) is measured from the centroid.

Fig. 5-5. Shaded area used in deriving the parallel-axis theorem

\[ I_{zz} = \int_A (d + y)^2 \, dA \]

where \( d \) is the distance from the centroid to the area. Squaring the quantities in the parentheses and placing the constants outside the integral signs

\[ I_{zz} = d^2 \int_A dA + 2d \int_A y \, dA + \int_A y^2 \, dA = Ad^2 + 2d \int y \, dA + I_o \]

However, since the axis from which \( y \) is measured passes through the centroid of the area, \( \int y \, dA \) or \( \bar{y}A \) is zero. Hence

\[ I_{zz} = I_o + Ad^2 \quad \text{(5-2)} \]

This is the parallel-axis theorem. It can be stated as follows: The moment of inertia of an area around an axis is equal to the moment of inertia of the same area around a parallel axis passing through the area’s centroid, plus

*By definition the principal axes are those about which the rectangular moment of inertia is a maximum or a minimum. Such axes are always mutually perpendicular. The product of inertia, defined by \( \int xy \, dA \) vanishes for the principal axes. An axis of symmetry of a cross-section is always a principal axis. For further details see the appendix to Chapter 8.
the product of the same area and the square of the distance between the two axes.*

The following examples illustrate the method of computing \( I \) directly by integration for two simple areas. Then an application of the parallel-axis theorem to a composite area is given. Values of \( I \) for commercially fabricated steel beams, angles, and pipes are given in Tables 3 to 8 of the Appendix.

**EXAMPLE 5-1**

Find the moment of inertia around the horizontal axis passing through the centroid for the rectangular area shown in Fig. 5-6.

**SOLUTION**

The centroid of this section lies at the intersection of the two axes of symmetry. Here it is convenient to take \( dA \) as \( b \ dy \). Hence

\[
I_{zz} = I_0 = \int_A y^2 \ dA = \int_{-h/2}^{h/2} y^2 b \ dy = b \left[ \frac{y^3}{3} \right]_{-h/2}^{h/2} = \frac{bh^3}{12} \tag{5-3}
\]

Similarly

\[
I_{yy} = \frac{b^3 h}{12}
\]

These expressions are used frequently, as rectangular beams are commonly employed in practice.

**EXAMPLE 5-2**

Find the moment of inertia about a diameter for a circular area of radius \( c \), Fig. 5-7.

**SOLUTION**

Since there is some chance of confusing \( I \) with \( I_\rho \) for a circular section, it is well to refer to \( I \) as the *rectangular* moment of inertia of the area in this case.

To find \( I \) for a circle, first note that \( \rho^2 = z^2 + y^2 \), as may be seen from the figure. Then using the definition of \( J \), noting the symmetry around both axes, and using Eq. 3-2

\[
I_\rho = \int_A \rho^2 \ dA = \int_A \left( y^2 + z^2 \right) \ dA = \int_A y^2 \ dA + \int_A z^2 \ dA
\]

\[
= I_{xx} + I_{yy} = 2I_{zz}
\]

\[
I_{xx} = I_{yy} = \frac{J}{2} = \frac{\pi c^4}{4} \tag{5-4}
\]

* A similar theorem can be proved for the product of inertia (for definition of the product of inertia see the appendix to Chapter 8): \( I_{yz} = I_{zy} = Ad_1 \) where \( I_{yz} \) is the product of inertia of an area \( A \) about the centroidal axis \( y \) and \( z \) and \( I_{zy} \) is the product of inertia of the same area about a set of parallel axes \( y, z \). The distance between \( y \) and \( y \) is \( d_1 \), and that between \( z \) and \( z \) is \( d_2 \). Subsequently this relation will be referred to as Eq. 5-2a.
In mechanical applications circular shafts often act as beams, hence Eq. 5-4 will be found useful. For a tubular shaft, the moment of inertia of the hollow interior must be subtracted from the above expression.

**EXAMPLE 5-3**

Determine the moment of inertia \( I \) around the horizontal axis for the area shown in Fig. 5-8 for use in the flexure formula.

**SOLUTION**

As the moment of inertia wanted is for use in the flexure formula, it must be obtained around the axis through the centroid of the area. Hence the centroid of the area must be found first. This is most easily done by treating the entire outer section and deducting from it the hollow interior. For convenience, the work is carried out in tubular form. Then the parallel-axis theorem is used to obtain \( I \).

![Diagram of the area with dimensions](image)

(All dimensions in mm)  
Fig. 5-8.

<table>
<thead>
<tr>
<th>Area</th>
<th>( A ) [mm²]</th>
<th>( y ) [mm]</th>
<th>( A\bar{y} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entire area</td>
<td>40(60) = 2400</td>
<td>30</td>
<td>72 000</td>
</tr>
<tr>
<td>Hollow interior</td>
<td>-20(30) = -600</td>
<td>35</td>
<td>-21 000</td>
</tr>
<tr>
<td>( \sum A ) = 1800 mm²</td>
<td>( \sum A\bar{y} = 51 000 \text{ mm}^3 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\bar{y} = \frac{\sum A\bar{y}}{\sum A} = \frac{51 000}{1800} = 28.3 \text{ mm from bottom}
\]

For entire area:  
\[
I_y = \frac{bh^3}{12} = \frac{40(60)^3}{12} = 7.2 \times 10^4 \text{ mm}^4
\]

\[
Ad^2 = 2400(30 - 28.3)^2 = 0.69 \times 10^4 \text{ mm}^4
\]

\[I_y = 72.69 \times 10^4 \text{ mm}^4\]

For hollow interior:  
\[
I_y = \frac{bh^3}{12} = \frac{20(30)^3}{12} = 4.50 \times 10^4 \text{ mm}^4
\]

\[
Ad^2 = 600(35 - 28.3)^2 = 2.69 \times 10^4 \text{ mm}^4
\]

\[I_y = 7.19 \times 10^4 \text{ mm}^4\]

For composite section:  
\[
I_y = (72.69 - 7.19) \times 10^4 = 65.50 \times 10^4 \text{ mm}^4
\]

Note particularly that in applying the parallel-axis theorem, each element of the composite area contributes two terms to the total \( I \). One term is the moment of inertia of an area around its own centroidal axis,
the other term is due to the transfer of its axis to the centroid of the whole area. Methodical work is the prime requisite in solving such problems correctly.

5-6. REMARKS ON THE FLEXURE FORMULA

The bending stress at any point of a beam section is given by Eq. 5-1a, \( \sigma = -\frac{M y}{I} \). The largest stress at the same section follows from this relation by taking \( |y| \) at a maximum, which leads to Eq. 5-1 \( \sigma_{\text{max}} = \frac{Mc}{I} \). In most practical problems the maximum stress given by Eq. 5-1 is the quantity sought; thus it is desirable to make the process of determining \( \sigma_{\text{max}} \) as simple as possible. This can be accomplished by noting that both \( I \) and \( c \) are constants for a given section of a beam. Hence \( I/c \) is a constant. Moreover, since this ratio is only a function of the cross-sectional dimensions of a beam, it can be uniquely determined for any cross-sectional area. This ratio is called the elastic section modulus of a section and will be designated by \( S \). With this notation Eq. 5-1 becomes

\[
\sigma_{\text{max}} = \frac{Mc}{I} = \frac{M}{\frac{I}{c}} = \frac{M}{S}
\]

(5-5)

or stated otherwise

maximum bending stress = \( \frac{\text{bending moment}}{\text{elastic section modulus}} \)

If the moment of inertia \( I \) is measured in \( m^4 \) and \( c \) in \( m \), \( S \) is measured in \( m^2 \). Likewise, if \( M \) is measured in newton-meters, the units of stress, as before, become newtons per square meter (N/m²). It bears repeating that the distance \( c \) as used here is measured from the neutral axis to the most remote fiber of the beam. This makes \( I/c = S \) a minimum, and consequently \( M/S \) gives the maximum stress. The efficient sections for resisting bending have as large an \( S \) as possible for a given amount of material. This is accomplished by locating as much of the material as possible far from the neutral axis.

The use of the term elastic section modulus in Eq. 5-5 corresponds somewhat to the use of the area term \( A \) in Eq. 1-1 (\( \sigma = P/A \)). However, only the maximum flexural stress on a section is obtained from Eq. 5-5, but the stress computed from Eq. 1-1 holds true across the whole section of a member.

Equation 5-5 is widely used in practice because of its simplicity. To facilitate its use, section moduli for many manufactured cross sections are tabulated in handbooks. Values for a few steel sections are given in Tables 3 to 8 in the Appendix. Equation 5-5 is particularly convenient for the design of beams. Once the maximum bending moment for a beam is determined and an allowable stress is decided upon, Eq. 5-5 may be solved for the required section modulus. This information is sufficient to select a beam.
However, a detailed consideration of beam design will be delayed until Chapter 10. This is necessary inasmuch as a shearing force, which in turn causes stresses, usually also acts at a beam section. The interaction of the various kinds of stresses must be considered first to gain complete insight into the problem.

The application of the flexure formulas to particular problems should cause little difficulty if the meaning of the various terms occurring in them has been thoroughly understood. The following two examples illustrate investigations of bending stresses at specific sections.

**EXAMPLE 5-4**

A 0.3 m by 0.4 m wooden cantilever beam weighing 76 kg/m carries an upward concentrated force of 20 kN at the end, as shown in Fig. 5-9(a). Determine the maximum bending stresses at a section 2 m from the free end.

![Diagram](image)

**SOLUTION**

A free-body diagram for a 2 m segment of the beam is shown in Fig. 5-9(c). To keep this segment in equilibrium requires a shear of \(20 - 0.75(2) = 18.5\) kN and a bending moment of \(20(2) - 0.75(2)1 = 38.5\) kN\(\cdot\)m at the cut section. Both of these quantities are shown with their proper sense in Fig. 5-9(c).

By inspecting the cross-sectional area, the distance from the neutral axis to the extreme fibers is seen to be 0.2 m, hence \(c = 0.2\) m. This is applicable to both the tension and the compression fibers.

From Eq. 5-3: \(I_{zz} = \frac{bh^3}{12} = \frac{(0.3)(0.4)^3}{12} = 16 \times 10^{-4} \text{ m}^4\)

From Eq. 5-1: \(\sigma_{\text{max}} = \frac{Mc}{I} = \frac{(38.5)(0.2)}{16 \times 10^{-4}} = \pm 4813 \text{ kN/m}^2\)
From the sense of the bending moment shown in Fig. 5-9(c) the top fibers of the beam are seen to be in compression, and the bottom ones in tension. In the answer given, the positive sign applies to the tensile stress, the negative sign applies to the compressive stress. Both of these stresses decrease at a linear rate toward the neutral axis where the bending stress is zero. The normal stresses acting on infinitesimal elements at A and B are shown in Fig. 5-9(d). It is important to learn to make such a representation of an element as it will be frequently used in Chapters 7, 8, and 9.

**ALTERNATE SOLUTION**

If only the maximum stress is desired, the equation involving the section modulus may be used. The section modulus for a rectangular section in algebraic form is

\[ S = \frac{I}{c} = \frac{bh^3}{12} \frac{2}{h} = \frac{bh^2}{6} \]  \hspace{1cm} (5-6)

In this problem, \( S = (0.3)(0.4)^2/6 = 8 \times 10^{-3} \text{ m}^3 \), and by Eq. 5-5

\[ \sigma_{\text{max}} = \frac{M}{S} = \frac{38.5}{8 \times 10^{-3}} = 4813 \text{ kN/m}^2 \text{ or kPa.} \]

Both solutions lead to identical results.

**EXAMPLE 5-5**

Find the maximum tensile and compressive stresses acting normal to the section A-A of the machine bracket shown in Fig. 5-10(a) caused by the applied force of 35 kN.

**SOLUTION**

The shear and bending moment of proper magnitude and sense to maintain the segment of the member in equilibrium are shown in Fig. 5-10(c). Next the neutral axis of the beam must be located. This is done by locating the centroid of the area shown in Fig. 5-10(b) (see also Fig. 5-10(d)). Then the moment of inertia about the neutral axis is computed. In both these calculations the legs of the cross section are assumed rectangular, neglecting fillets. Then, keeping in mind the sense of the resisting bending moment and applying Eq. 5-1, one obtains the desired values.

<table>
<thead>
<tr>
<th>Area Number</th>
<th>( A ) [mm²]</th>
<th>( y ) [mm] (from ( ab ))</th>
<th>( Ay ) [( \times 10^3 \text{ mm}^3 )]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2500</td>
<td>12.5</td>
<td>31.3</td>
</tr>
<tr>
<td>2</td>
<td>1875</td>
<td>62.5</td>
<td>117</td>
</tr>
<tr>
<td>3</td>
<td>1875</td>
<td>62.5</td>
<td>117</td>
</tr>
<tr>
<td>( \sum A = 6250 \text{ mm}^2 )</td>
<td>( \sum Ay = 265.3 \times 10^3 \text{ mm}^3 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5-9(c) the top ones in tensile stress, these stresses ending stress is A and B are representation of 9.

(5-6)

:q. 5-5
:Pa.

:mg normal to the a) caused by the

sense to maintain Fig. 5-10(c). Next ne by locating the 5-10(d). Then the both these calcula-

cation and apply-

\[
\bar{y} = \frac{\sum A_y}{\sum A} = \frac{265.3 \times 10^3}{6250} = 42.5 \text{ mm from the line } ab
\]

\[
I = \sum (I_o + Ad^2) = \frac{(100)(25)^3}{12} + (2500)(30)^2 + \frac{(2)(25)(75)^3}{12} + (2)(1875)(20)^2
\]

\[
= 5.64 \times 10^4 \text{ mm}^4
\]

\[
\sigma_{\text{max}} = \frac{Mc}{I} = \frac{35 \times 400 \times 57.5}{5.64 \times 10^6} = 0.143 \text{ kN/mm}^2 = 143 \text{ MPa (compression)}
\]

\[
\sigma_{\text{max}} = \frac{Mc}{I} = \frac{35 \times 400 \times 42.5}{5.64 \times 10^6} = 0.105 \text{ kN/mm}^2 = 105 \text{ MPa (tension)}
\]

These stresses vary linearly toward the neutral axis and vanish there. If for the same bracket the direction of the force P were reversed, the sense of the above stresses would also reverse. The results obtained would be the same if the cross-sectional area of the bracket were made T-shaped as shown in Fig. 5-10(e). The properties of this section about the significant axis are the same as those of the channel. Both these sections have an axis of symmetry.

The above example shows that members resisting flexure may be proportioned so as to have a different maximum stress in tension than in compression. This is significant for materials having different strengths in tension and compression. For example, cast iron is strong in compression and weak in tension. Thus, the proportions of a cast-iron member may be so