Inflating a Rubber Balloon

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Abstract

A spherical balloon has a non-monotonic pressure-radius characteristic. This fact leads to interesting stability properties when two balloons of different radii are interconnected, see [1], [2], [3]. Here, however, we investigate what happens when a single balloon is inflated by mouth (say). We simulate that process and show how the maximum of the pressure-radius characteristic is overcome by the pressure in the lungs and how the downward sloping part of the characteristic is "bridged" while the lung pressure relaxes.

Keywords: Rubber balloons, Mooney-Rivlin material,
Non-convexity, Stability.

1 Characteristic of a spherical balloon

The $([p_B], r)$-characteristic, which dictates the dependence of the pressure jump $[p_B]$ across the membrane of a spherical rubber balloon on its radius r, is non-monotonic, see [1] and Fig.1. If the stress-strain relation of rubber is of the Mooney-Rivlin type, the analytic form of the $([p_B], r)$-relation reads

$$[p_B](r) = 2s_1 \frac{d_0}{r_0}\left(\frac{r_0}{r} - \left(\frac{r_0}{r}\right)^7\right)\left(1 - \frac{s_1}{s_{-1}}\left(\frac{r}{r_0}\right)^2\right). \tag{1.1}$$

d_0 and r_0 are the thickness and the radius of the balloon, respectively, before inflation, and s_1 and s_{-1} are the two constants of a Mooney-Rivlin material. For a typical rubber balloon we have

$$s_1 = 3\ bar,\ s_{-1} = -0.3\ bar, \text{ and } \frac{d_0}{r_0} = 0.5 \cdot 10^{-2}. \tag{1.2}$$

For brevity we introduce $K = -\dfrac{s_1}{s_{-1}} = 10$.

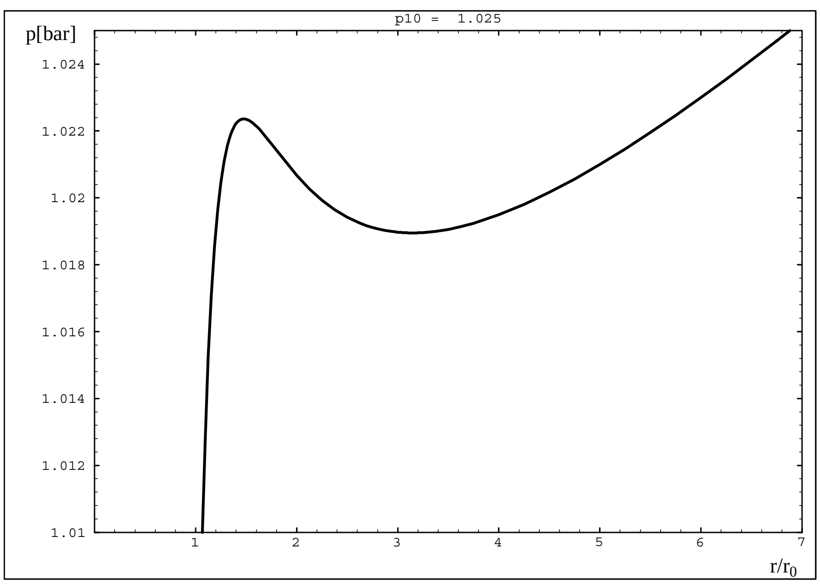

Fig. 1 (pressure, radius)-characteristic.

The free energy F_B of the balloon results from integration

$$F_B = \int_{r_0}^{r} [p_B]\, 4\pi r^2 dr \tag{1.3}$$

$$F_B = \frac{3}{2}\left(s_1 \frac{d_0}{r_0}\right) V_{B0}\left[2\left(\frac{r}{r_0}\right)^2 + \left(\frac{r_0}{r}\right)^4 - 3 + K\left(\left(\frac{r}{r_0}\right)^4 + 2\left(\frac{r_0}{r}\right)^2 - 3\right)\right],$$

where $V_{B0} = \frac{4\pi}{3} r_0^3$ is the volume contained in the balloon before inflation.

The question arises of how the part with negative slope is traversed as we inflate the balloon. In order to obtain an answer we consider a model which, in our understanding, simulates the inflation of a balloon by mouth.

2 Modelling inflation

Fig. 2 shows a schematic view of our "inflation apparatus". It consists of the balloon, a cylinder with piston of cross section F, a linearly elastic spring, and two valves A and B. The volume of the cylinder represents the volume of the lungs and the force in the spring stands for the muscle forces that push the air into the balloon.

Inflation usually occurs in several steps $i = 1, 2, ...$, of which each one has four phases, viz.

i_1: "Inhaling". We begin the ith step with a balloon of radius r_{i-1}. Valve B is closed and valve A is open; the spring is unloaded and the initial volume of the cylinder is V_{ZA}. That volume is increased by lifting the piston so that the volume becomes $V_{Z\max}$. Then valve A is closed.

i_2: "Pressurizing". The piston is released so that the air in the cylinder is compressed by the spring to the volume V_{Z0}. The value of the pressure is then called P.

i_3: "Inflating". Upon opening valve B the compressed air will enter the balloon, which increases to the radius r_i with the corresponding pressure p_i.

i_4: "Changing pressure". Valve B is closed and valve A is opened so that the pressure p_{Zi} in the cylinder drops to the external pressure p_0. The process is then repeated.

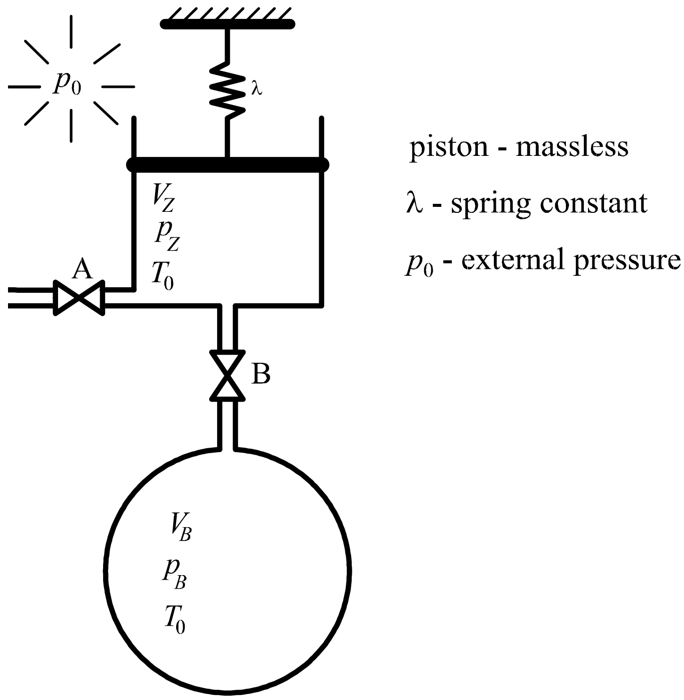

Fig. 2 Model for lung and balloon.

3 Equilibria

It is our objective to calculate the radii r_i for a prescribed pressure P, or a prescribed spring constant λ. These are the radii for which $-$ at the end of the phase i_3 $-$ the system of spring, cylinder and balloon are in equilibrium. The condition for the equilibrium is the existence of a minimum of the available free energy. In the present case that energy has the form, see [3]

$A = N_Z kT \ln \frac{p_Z}{p_0} + N_B kT \ln p_B \frac{p_B}{p_0} + (N_Z + N_B)\, a\,(T, p_0) +$	free energy of the air in Z and B
$+\frac{3}{2}\left(s_1 \frac{d_0}{r_0}\right) V_{B0} \Big[2\left(\frac{r}{r_0}\right)^2 + \left(\frac{r_0}{r}\right)^4 - 3$ $\quad + K\left(\left(\frac{r}{r_0}\right)^4 + 2\left(\frac{r_0}{r}\right)^2 - 3\right)\Big]$	free energy of the balloon
$+ p_0(V_Z + V_B)$	work of external pressure p_0
$+ \frac{\lambda}{2F^2}(V_Z - V_{ZA})^2$.	energy of the spring.

(3.1)

a is the specific free energy of the air in the reference state (T, p_0); it is a constant.

The pressures p_B and p_Z are related to N_B and V_B, or N_Z and V_Z, respectively, by the ideal gas relation $pV = NkT$. Therefore the available free energy is a function of N_B, (N_Z), $V_B = \frac{4\pi}{3} r^3$ and V_Z. The total number $N = N_B + N_Z$ of molecules is constant during the phases i_3 of inflation, but it depends on i. Indeed, we have

$$N_i kT = PV_{Z0} + p_{B(i-1)} \frac{4\pi}{3} r_{i-1}^3, \tag{3.2}$$

so that the number N_i equals the sum of the $-$ always equal $-$ cylinder filling and of the balloon filling reached in the $(i-1)^{st}$ step.

A necessary condition for equilibria requires that the derivatives of A with respect to N_B, r, and V_Z vanish. From this condition we obtain easily

$p_B = p_Z$	pressure in balloon = pressure in cylinder
$p_Z - p_0 = \frac{\lambda}{F^2}(V_Z - V_{ZA})$	pressure jump at piston = spring pressure
$p_B - p_0 = [p_B]\ (r)$	pressure jump at balloon = membrane pressure.

(3.3)

These are three equations for the equilibrium values of N_B, r, and V_Z in each step of inflation. Each one of these values depends on the step number i, because we have $p_Z V_Z = (N_i - N_B)kT$.

The solution of (3.3) must be found numerically. There are several solutions which are not all stable. In a stable equilibrium the matrix of second derivatives of the available free energy A in (3.1) with respect to N_B, r, and V_Z must be positive definite. That is a sufficient condition for a minimum of A. The exploitation of the condition, however, is extremely cumbersome and therefore we proceed differently.

4 The pressure equilibrium between cylinder and balloon

We assume that the equilibria $(3.3)_{2,3}$ of piston and membrane are established so quickly that the slower trend to establish the equilibrium $(3.3)_1$ between cylinder and balloon always sees $(3.3)_{2,3}$ satisfied. If that is so, we may use $(3.3)_{2,3}$ to determine $p_Z(r)$ and $p_B(r)$. We obtain

$$\begin{aligned}
&p_Z(r) - p_0 = \tfrac{\lambda}{F^2}\left(V_Z(r) - V_{ZA}\right) \quad \text{and} \\
&p_B(r) - p_0 = 2s_1 \tfrac{d_0}{r_0}\left(\tfrac{r_0}{r} - \left(\tfrac{r_0}{r}\right)^7\right)\left(1 - \tfrac{s_1}{s_{-1}}\left(\tfrac{r}{r_0}\right)^2\right), \quad \text{with} \\
&V_Z(r) = -\tfrac{1}{2}\left(\tfrac{p_0 F^2}{\lambda} - V_{ZA}\right) + \sqrt{\tfrac{1}{4}\left(\tfrac{p_0 F^2}{\lambda} - V_{ZA}\right)^2 + \tfrac{F^2}{\lambda}\left(N_i kT - p_B(r)\tfrac{4\pi}{3}r^3\right)}\,.
\end{aligned} \tag{4.1}$$

The equation $(4.1)_3$ for $V_Z(r)$ follows from $(4.1)_1$ with

$$p_Z(r) = \frac{1}{V_Z}\left(N_i - N_B\right)kT = \frac{1}{V_Z}\left(N_i kT - p_B(r)\frac{4\pi}{3}r^3\right)$$

as the solution of a quadratic equation. By $(4.1)_{1,3}$ the function $p_Z(r) - p_0$ determines an ensemble of curves parametrized by N_i or, equivalently, r_{i-1}. Note that, by (3.2), there is a one-to-one correspondence between r_{i-1} and N_i, since $p_B(r)\frac{4\pi}{3}r^3$ is monotonic.

Fig. 3 shows that ensemble of curves, each one in the interval $r_{i-1} < r < r_i$. All individual curves $\overset{(i)}{p_Z}(r) - p_0$ begin at the height $P - p_0$. In the first step we have $i = 1$ and r_{i-1} equals r_0, the radius of the uninflated balloon. The first step ends at r_1 where $\overset{(1)}{p_Z}(r_1)$ intersects the curve $p_B(r)$. Vertically above that point at the height $P - p_0$ the curve $\overset{(2)}{p_Z}(r)$ starts and it runs through to r_2 where it intersects the curve $p_B(r)$, etc. Thus we see the zig-zag curves of Fig. 3 and 4 appear. The vertical branches represent the inhaling and pressurizing step with the closed valve B, while the arcs represent the inflating step. Equilibria exist in the lower tips where the arcs touch the balloon characteristic $p_B(r) - p_0$. In Fig. 3 we observe how much effort it may take to overcome the pressure maximum of that characteristic, when P is only slightly higher than the barrier. But the labor is rewarded, once the barrier is overcome because afterwards the balloon inflates in a single step with decreasing pressure to obtain a much bigger radius than the one with which it began.

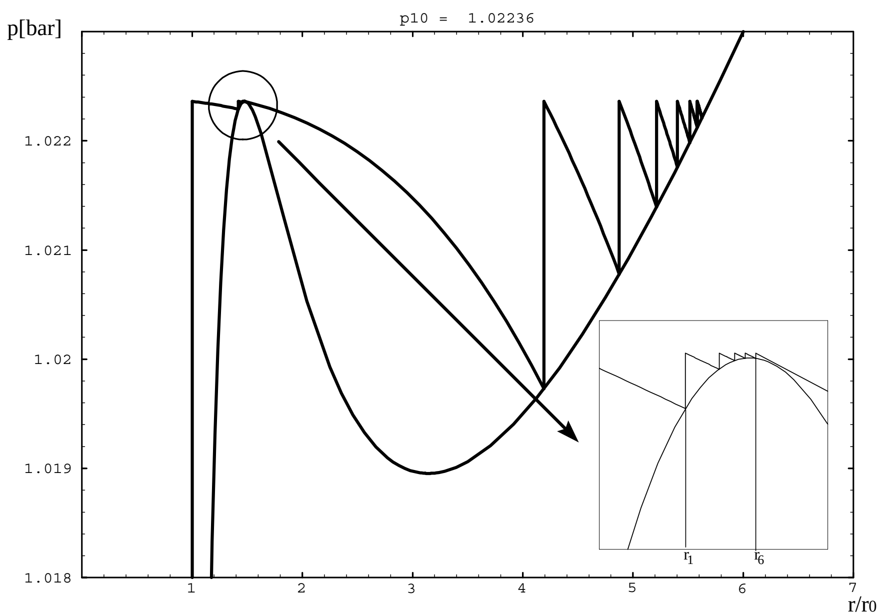

Fig. 3 Zig-zag line: Cylinder pressure during inhaling, pressurizing and inflating for a pressure that is minimally larger than the pressure barrier.
Smooth line: Pressure-radius characteristic of the balloon.

Fig. 4 shows the same process with the difference that P is now large, so that a strong lung is at work. The pressure barrier of the balloon is overcome in the first step.

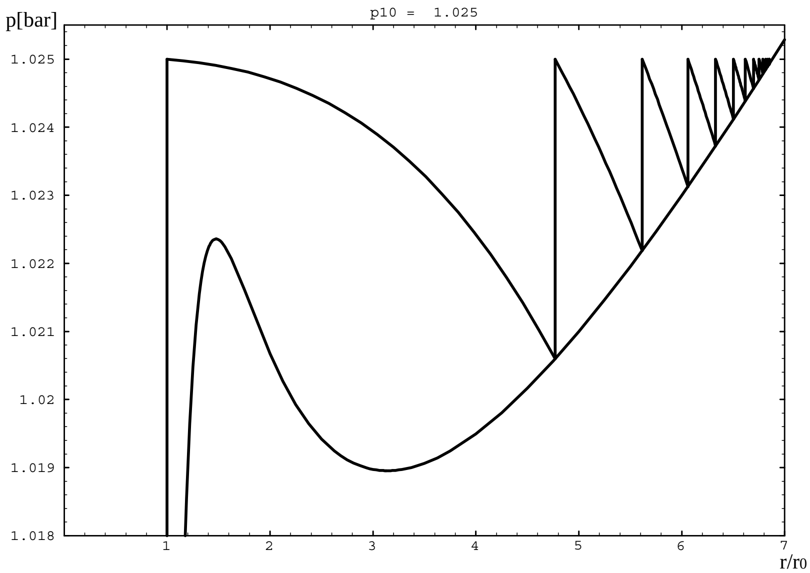

Fig. 4 Zig-zag curve shows the pressure in the cylinder, when the maximal pressure P is much bigger than the pressure barrier. Equilibria exist in the lower tips.

The data for which the Figs. 3 through 6 are drawn were chosen as follows.

$$s_1 \frac{d_0}{r_0} = 1,5 \cdot 10^3 \frac{N}{m^2} \qquad K = 10 \qquad p_0 = 1 \text{ bar} \qquad T = 290\ K$$

$$V_{B0} = 10^{-6} m^3, \qquad V_{ZA} = 4 \cdot 10^{-3} m^3, \qquad V_{Z0} = 4,5 \cdot 10^{-3} m^3$$

$$\frac{\lambda}{F^2} = \left[\begin{array}{lcll} 2,236 \cdot 10^6 \ \frac{N}{m} \frac{1}{m^4} & & \text{Fig. 3,5,6} & \Longrightarrow \quad P = \ 1,02236\ bar \\ & \text{for} & & \\ 2,5 \cdot 10^6 \ \frac{N}{m} \frac{1}{m^4} & & \text{Fig. 4} & \quad P = \ 1,025\ bar \end{array} \right.$$

5 Available free energy as a function of r

We continue to consider the partial equilibria in which only the equilibrium condition $(3.3)_1$ is not yet satisfied, while the conditions $(3.3)_{2,3}$ are already satisfied. In that case we may write the available free energy A in (3.1) as a function of r. We obtain

$$\begin{aligned} A - N_i a(T,p_0) = {} & p_Z V_Z \ln\frac{p_Z}{p_0} + p_B \frac{4\pi}{3} r^3 \ln\frac{p_B}{p_0} + \\ & + \frac{3}{2}\left(s_1\frac{d_0}{r_0}\right) V_{B0}\left[2\left(\frac{r}{r_0}\right)^2 + \left(\frac{r_0}{r}\right)^4 - 3 + K\left(\left(\frac{r}{r_0}\right)^4 + 2\left(\frac{r_0}{r}\right)^2 - 3\right)\right] + \\ & + p_0\left(V_Z + \frac{4\pi}{3} r^3\right) + \\ & + \frac{\lambda}{2F^2}\left(V_Z - V_{ZA}\right)^2 , \end{aligned} \tag{5.1}$$

where $p_Z(r)$ and $p_B(r)$ as well as $V_Z(r)$ are given by (4.1).

$A(r)$ is drawn in the lower part of Fig.5 for the second step of the inflation process and for the small pressure P to which Fig.3 refers. $A(r)$ has three extrema corresponding to the three points of intersection of the curves $\overset{(2)}{p_Z}(r)$ and $p_B(r)$, see upper part of Fig. 5.

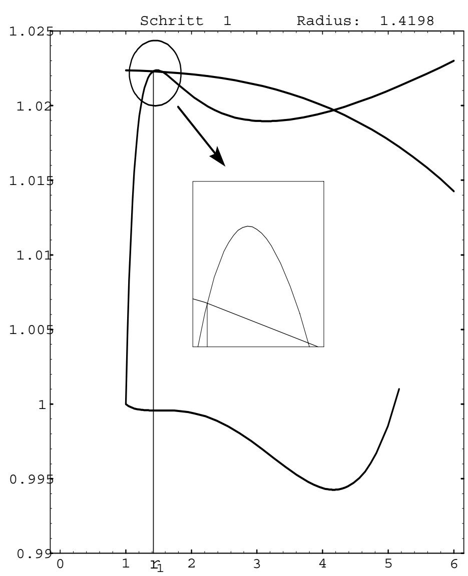

Fig. 5 $A(r)$ with two minima and three points of intersection of the p-curves.

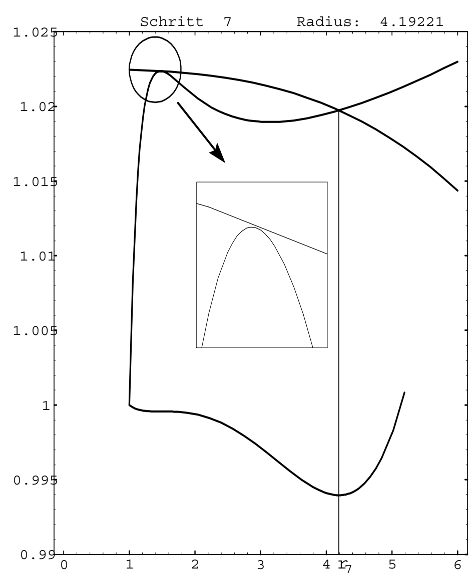

Fig. 6 $A(r)$ with one minimum and one point of intersection of the p-curves.

[In Fig. 3 we have not seen these three intersections, since we have cut off the curve $\overset{(2)}{p}_Z(r)$ at the first point of intersection.] The central extremum is a maximum and therefore corresponds to an unstable state. The other two extrema are minima and therefore they represent stable states. Starting from r_{i-1} the balloon will find the *nearest* minimum with $r_i > r_{i-1}$, since it cannot overcome the energetic barrier. In the seventh step the left minimum, $-$ and the maximum $-$ have been eliminated. The p-curves have only one point of intersection, see Fig. 6 and the balloon expands strongly.

6 Discussion

Rubber as such and, in particular the material of rubber balloons is not strictly a Mooney-Rivlin material. There are semiempirical formulae that fit the experimental (p, r)-curves better, e.g. see [4], [5], [6]. A peculiarity of these improved constitutive relations is that the balloons may lose the spherical symmetry at a certain radius. This interesting aspect of balloon physics does not show up here, since we treat rubber as a Mooney-Rivlin material. We do mention in this context the expert review on hyperelasticity of rubbers $-$ among other topics $-$ by M.F. Beatty [7]. An interesting work on non-spherical balloons may also be found in [8].

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